Moving Bottlenecks: A Numerical Method that Converges in Flows

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ABSTRACT

This paper presents a numerical method to model kinematic wave (KW) traffic streams containing slow vehicles. The slow vehicles are modeled discretely as moving boundaries that can affect the traffic stream. The proposed scheme converges in flows, densities and speeds without oscillations, and therefore can be readily used in situations where one wishes to model the effect of the traffic stream on the bottlenecks too. The approach is more accurate than Godunov's method in situations where the latter can be applied, i.e., without moving bottlenecks.

1. INTRODUCTION

This paper is a sequel to Daganzo and Laval (2003), which proposed a way to treat moving bottlenecks numerically within the context of kinematic wave (KW) theory. This reference showed that if the moving bottlenecks are replaced by sequences of fixed bottlenecks that are restricted to be on a space-time lattice, then the difference between the exact and approximate vehicle counts at any point in space-time are uniformly bounded by a quantity that tends to zero as the lattice spacing is reduced. Good features of this procedure are: (i) if exact solution methods are used to solve the approximate problem, the solution error in vehicle count is uniformly bounded; and (ii) "off-the-shelf" software can be used to solve the approximate problems. The main disadvantage is that the approximate flows, densities and speeds at a point do not converge to the exact ones, even as vehicle counts do. Thus, to provide estimates of these quantities, one needs to average their values over finite regions; i.e., multiple cells. We present below a new numerical scheme that overcomes this disadvantage, albeit by doing away with good points (i) and (ii).

2. COMPOSITE-RIEMANN PROBLEMS

A building block of the proposed method is the solution of a class of initial value problems containing two moving bottlenecks, i and i+1, with linear trajectories. These problems will be called composite Riemann problems, or CRP's. The space-time geometry of a typical CRP and its input data are depicted in Fig. 1. Given are the following constants: (i) the locations of the bottlenecks at time zero (x_i and x_{i+1}); (ii) the initial vehicle numbers (N_i) at the location of the bottlenecks;[1] (iii) downstream and upstream densities (k_d and k_u); (iv) the speeds and maximum passing rates associated with each bottleneck (v_i, Q_i); and (v) a time of interest, Δt, such that bottleneck trajectories do not cross:

$$\Delta t \le (x_{i+1} - x_i)/(v_i - v_{i+1}); \quad \text{if} \quad v_i > v_{i+1} \qquad \text{(no crossing condition).} \qquad (1)$$

It is assumed that the initial density is uniform in the three intervals demarcated by the bottleneck positions. The density in the middle interval is $(N_i - N_{i+1})/(x_{i+1} - x_i)$. For maximum generality, bottlenecks are allowed to travel faster than the traffic stream—they do not have to be embedded in it. We then look for the vehicle numbers, $N_i(t)$ and $N_{i+1}(t)$, on the trajectories of the bottlenecks at time $t = \Delta t$.

This is a well-posed problem in kinematic wave theory. Its solution can be obtained with standard recipes. For problems with piecewise linear data, as is the case with the CRP, the exact solution can

[1] As is conventional, vehicles are assumed to travel in the direction of increasing x and to be numbered in decreasing order with x. Thus, the Moskowitz function of vehicle number $N(t, x)$ is non-decreasing in t at every x.

always be obtained with a finite number of calculations if the fundamental diagram is piece-wise linear. Appendix A shows a simple procedure that can be used when the flow-density relation is triangular.

3. THE SOLUTION METHOD

The general procedure works on a rectangular space-time lattice with a fixed spatial spacing, Δx, and variable time spacing Δt. We assume that at the current discrete time point we know the vehicle numbers at all the lattice points and bottleneck locations, that the density is constant between points and that the bottleneck speeds are also constant during the ensuing time step. The objective is predicting the vehicle numbers at the new bottleneck locations and all lattice points. The recipes of Sec. 2 do the job if we treat mesh lines as moving bottlenecks with zero speed and maximum passing rate equal to the road capacity, Q_{max}. We simply need to ensure that the time step is short enough to ensure that (1) applies (bottlenecks do not cross) and also that the domain of dependence for the "end-point" of bottleneck i includes at most one bottleneck root other than its own; i.e., that the new counts are CRP counts. If we let the range of feasible wave velocities be denoted $[-w, v_f]$, and also assume that bottleneck speeds are in the range, the CRP condition is:

$$\Delta t \le (x_{i+2} - x_i)/(w + v_i), \quad \Delta t \le (x_i - x_{i-2})/(v_f - v_i), \quad \text{and}$$
$$\Delta t \le \max\{(x_{i+1} - x_i)/(w + v_i)\,;\,(x_i - x_{i-1})/(v_f - v_i)\} \quad \text{for all } i. \qquad (2)$$

This condition ensures that bottleneck i interacts at most with one other companion bottleneck; either i or i+1. Thus, we propose choosing the largest Δt consistent with (1), (2) and with the assumption of constant bottleneck speed. An iteration of the procedure is as follows:

Step 1. Calculate the (uniform) densities between points i and i+1 with, $k_{i,i+1} = (N_i - N_{i+1})/(x_{i+1} - x_i)$; if $x_{i+1} = x_i$ put $k_{i,i+1} = 0$. Do for all i.

Step 2. For the known bottleneck speeds in the new interval, determine the new step size with (1) and (2), and the final bottleneck positions.

Step 3. Predict the new counts by solving a CRP for each i (with its appropriate companion, if any.)

Step 4. Update the bottleneck positions (from Step 2) and update the new (exogenous) maximum passing rates. If two bottlenecks coincide and will pass, reverse their indices. Update time. Stop or go to step 1.

4. DISCUSSION

The recipe proposed in this paper is numerically stable because it is the composition of two contraction mappings in the space of *N*-curves: an averaging operation (step 1) and the exact solution of a KW problem (steps 2-4); for more information on these mappings, see Daganzo (2001).

Notice from condition (2) that time steps can be longer than those of conventional methods, which have to satisfy Courant's condition. This implies that this procedure (i) avoids stalling when bottlenecks cross, and (ii) is more accurate in the absence of moving obstructions, as explained next.

The stalling problem arises when bottleneck interactions are not allowed and bottlenecks pass each other. Figure 2a shows how the series of time steps $\{t_i - t_{i-1}\}$ obtained with a no- the procedure performs in the same situation. The new procedure can only stall if three or more bottlenecks coincide at a point in space-time, or two bottlenecks coincide with a lattice line. This, however, can be easily avoided by locally perturbing the trajectory of the bottlenecks.

For problems without moving obstructions the proposed technique compares favorably with Godunov's method when $v_f > w$. To see this, recall that Godunov's method involves an averaging approximation similar to step 1, but it is performed twice as often because the step size is half of the one predicted by condition (2). Formally, this can be verified by evaluating the local truncation error for both methods and also by numerical experimentation. In particular, the proposed procedure is exact if the flow-density relation is triangular with $v_f = 2w$, whereas Godunov's method yields a significant truncation error.

The procedure can be used with inhomogeneous lattices and can be applied to inhomogeneous highways. The procedure is also well suited to deal with endogenous bottlenecks, since any reasonable rule can be used in step 4 to modify the maximum passing rates and in step 2 to modify the bottleneck trajectories; including rules that account for road geometry and the presence of other nearby bottlenecks.

4.1 Comparison with examples in Daganzo and Laval (2003)

To illustrate the properties of the new method, we ran the examples in Daganzo and Laval (2003) and compare the results vis a vis.

Figures 3 and 4 show the result of the method when applied to examples 1 and 2 of Daganzo and Laval (2003). These examples examine a one-mile homogeneous section of a two-lane freeway that obeys an isosceles fundamental diagram with free flow speed, v_f = 60 mph, jam density, k_j = 300 veh/mi, and wave velocity in congestion, $w = -v_f$. In example 1 the freeway is flowing at capacity when, at t_0 = .3 min and x = .3 mi, a truck instantaneously slows down to a constant speed v = 20 mph and then resumes free-flow speed at t = 2.1 min; see Fig. 3a. In example 2 the freeway is empty at t = 0, except for a lone truck at x = 0.3 mi that travels at a constant speed, v_1 = 11 mph; see Fig. 4a. At this time, a police car traveling at v_2 = 45 mph enters the road at x = 0, followed by a queue of cars. These examples assumed that trucks reduce the capacity by a factor of two and that cars do not pass the police car.

Part c of figures 3 and 4 show the density maps obtained with the method and this paper, which match part a of the figures quite accurately. Parts b and d depict the numerical *N*-values observed at the six evenly spaced detector stations in part c, using the method in Daganzo and Laval (2003) and the one presented here, respectively. Note the smooth flows in Fig. 3d compared to the oscillatory solution in part b; both with Δt = 3 secs. Close examination of Fig. 4d confirms that the proposed method introduces some numerical error in the *N*-values. This is due to the small time steps that need to be introduced when bottlenecks cross, which magnify the averaging errors of step 1. These errors can be reduced if one uses longer steps and CRP's with three-bottleneck interactions.

REFERENCES

Daganzo, C. F. (2001) "A simple traffic analysis procedure" *Nets. Sptl. Econ.* **1**, 77-101.

Daganzo, C.F. and Laval, J.A. (2003) "On the numerical treatment of moving bottlenecks," Institute of Transportation Studies, University of California, Berkeley, CA. (Submitted to *Trans. Res. B*).

Newell, G.F. (1993) A Simplified Theory of Kinematic Waves in Highway Traffic I: General Theory. II: Queuing at Freeway Bottlenecks. III. Multi-Destination Flows. *Trans. Res.* **27B**, 281-287.

APPENDIX :

Exact Solution of CRP's for Triangular Flow-Density Relations

The results in this appendix apply for problems with triangular flow-density relations with jam density k_j , free-flow speed v_f and wave speed $-w$. The proposed method is based on the following two facts, specific to CRP's:

Fact a. Consideration of all possible wave-maps reveals that in a CRP, there can be at most one flow change on each bottleneck. Further consideration reveals that if there is a singularity where the bottleneck changes status (from active to inactive or vice versa) the singularity must be at one of the four locations marked by dots in Figure A1. (The slopes of the slanted lines in this figure that are not bottleneck trajectories are either $-w$ or v_f ; they determine the location of the dots uniquely.)

Fact b. Consider now an arbitrary point "P" on one of the bottleneck trajectories. The vehicle number at this point can be obtained from the vehicle numbers at three anchor points {PU, PM, PD} (see figure) by:

$$N_P = min\{N_{PU} \,;\, N_{PD} + t_{PD}\, wk_j \,;\, N_{PM} + t_{PM}\, Q_i \} \qquad \text{(A1)}$$

if (as occurs in Fig. A1) the CRP anchors satisfy the following conditions: (1) PU is on the other bottleneck trajectory (or on the boundary) and a fast forward wave goes from PU to P without crossing any bottleneck trajectories; (2) PD is on the boundary (or on the other bottleneck trajectory) and a fast backward wave goes from PD to P in time t_{PD} without crossing any bottleneck trajectories; (3) the middle anchor is on the same bottleneck trajectory as P and the time separation between PM and P is t_{PM} ; and (4) all singularities in the domain of dependence of P are in the union of the domains of dependence of PU, PM and PD.

Fact (b) can be proven using Newell's minimum principle for triangular flow-density relations (Newell, 1993), modified to allow for homogeneous (constant-speed-constant-maximum-passing-rate) bottlenecks. The proof is beyond the scope of this note. Since each bottleneck can have at most one flow change, there can be at most three steady-state regimes. Therefore, it is possible to handle all the singularities with a three-step procedure. This procedure is described with reference to the point labels of Fig. A2:

Step1. Use (A1) to determine N_B, using {D,A,G} as anchors. The passing rate along AB is $(N_B - N_A)/t_B$ at all points. Repeat for segment DE, using as anchors {H,D,A}. The passing rate is also constant along DE. This determines $N_i(t)$ and $N_{i+1}(t)$ up to points E and B.

Step 2. Again, use (A1) to determine N_C and N_F, with{E, B, I} and {J, E, B} as anchors. The passing rates along segments BC and EF are again constant and equal to $(N_C - N_B)/(t_C - t_B)$ and $(N_F - N_E)/(t_F - t_E)$, respectively. Thus, $N_i(t)$ and $N_{i+1}(t)$ are now determined up to F and C.

Step 3. Using {F,C,K} and {L,F,C} as anchors, determine N_M and N_N, and the ensuing (constant) passing rates. This yields $N_i(t)$ and $N_{i+1}(t)$ from points F and C onward.

List of Figures

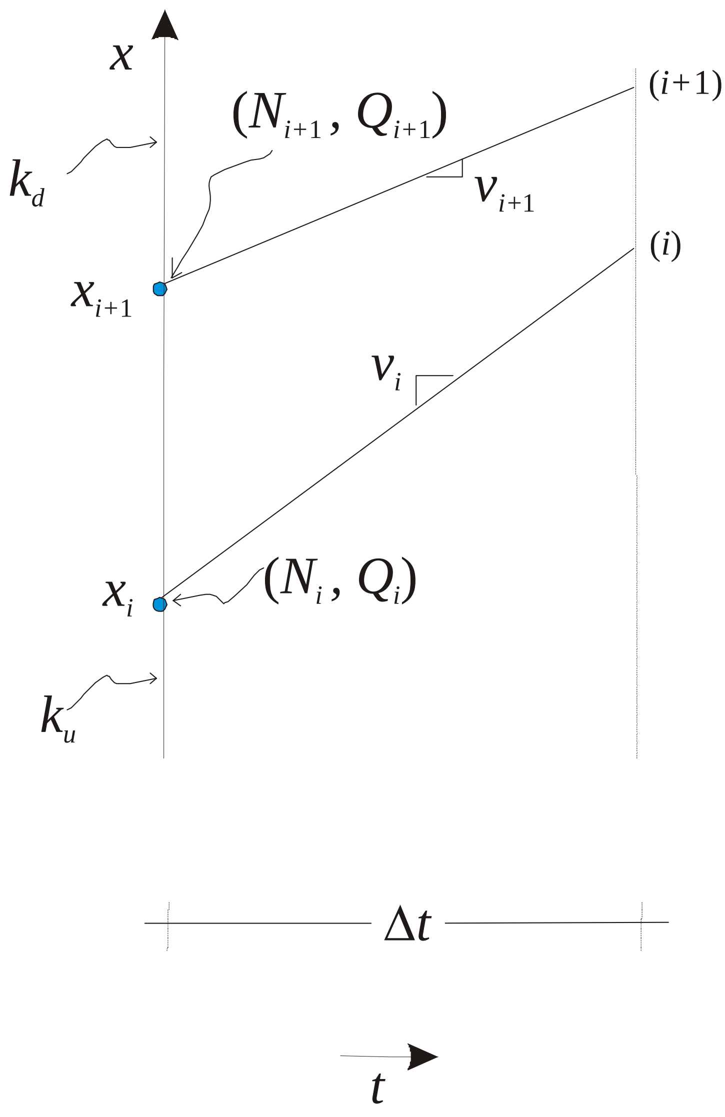

Figure 1: Data for a CRP

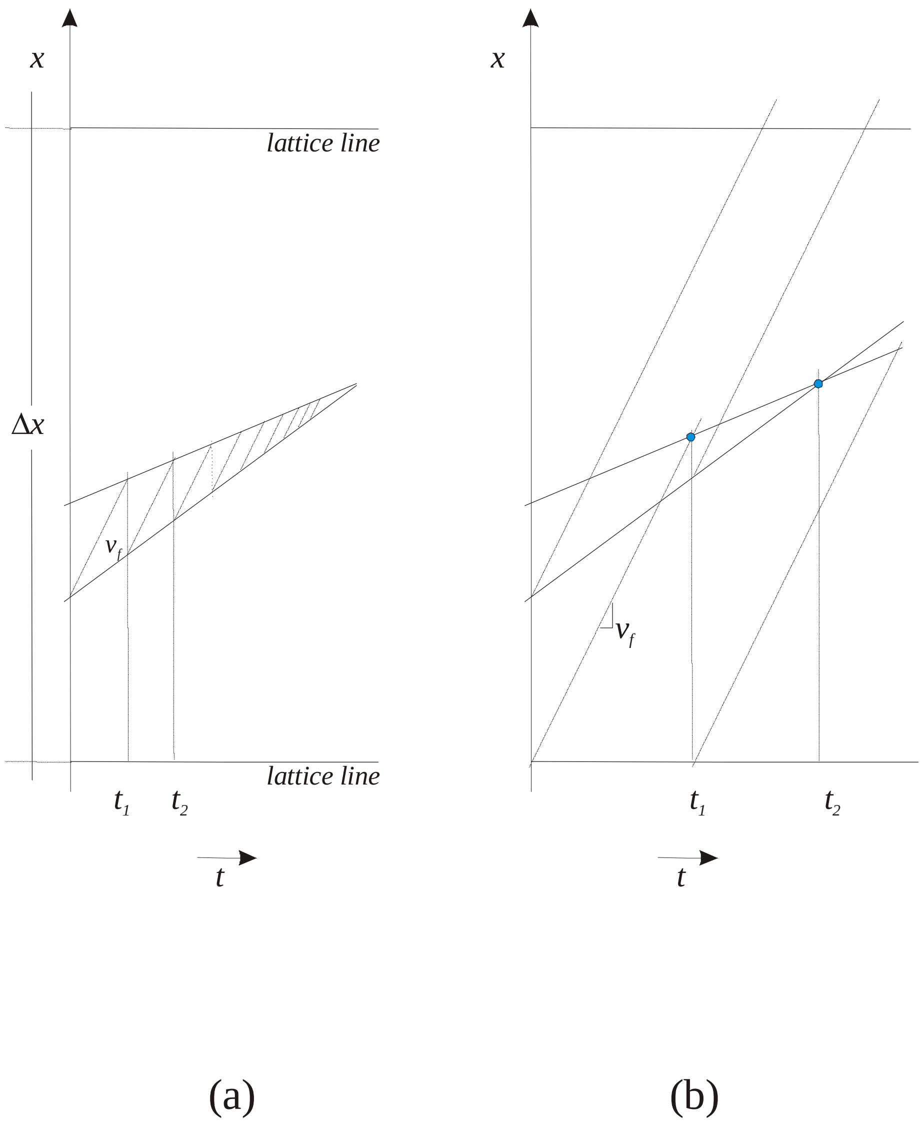

Figure 2: Time steps of two numerical procedures: (a) conventional; (b) proposed (assume $w << v_f$).

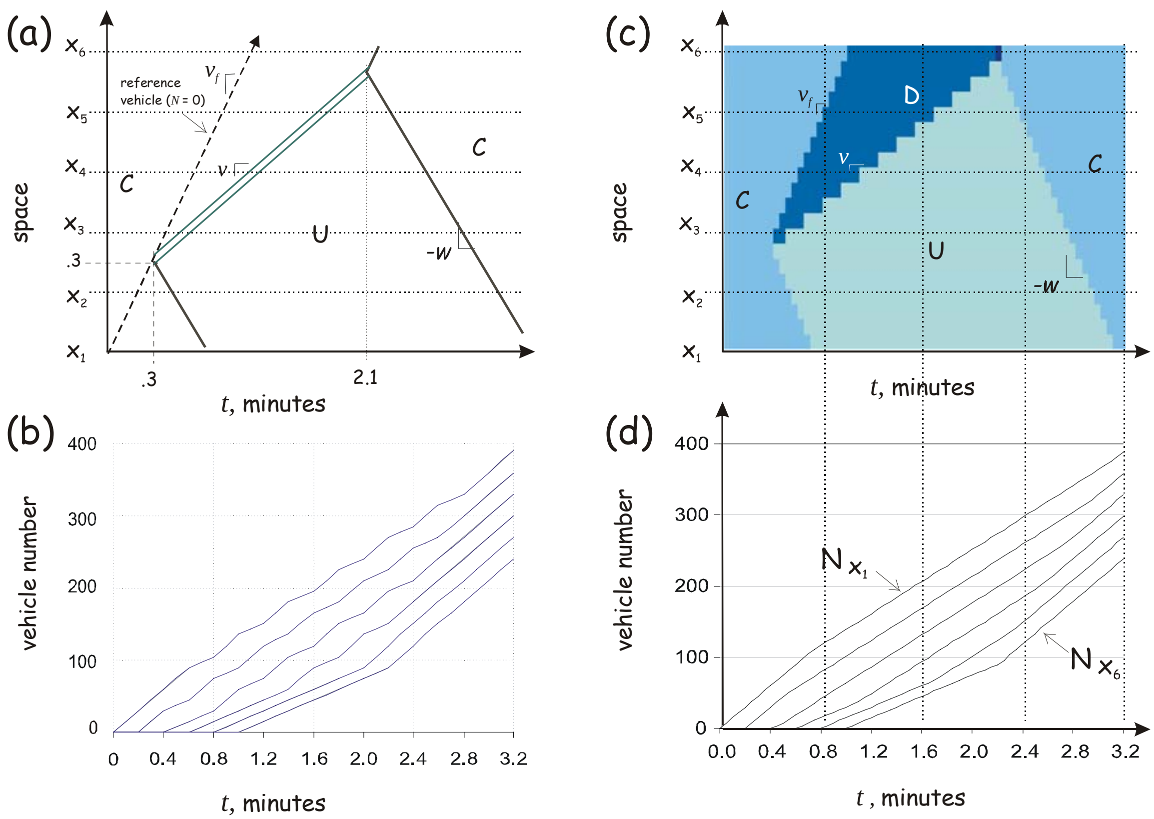

Figure 3: Comparison with example 1 in Daganzo and Laval 2003: (a) exact solution in the time-space diagram; (b) numerical *N*-curves with the method in Daganzo and Laval 2003 (Δt = 3 sec); (c) density map of the proposed method; (d) numerical *N*-curves with the proposed method (Δt = 3 sec).

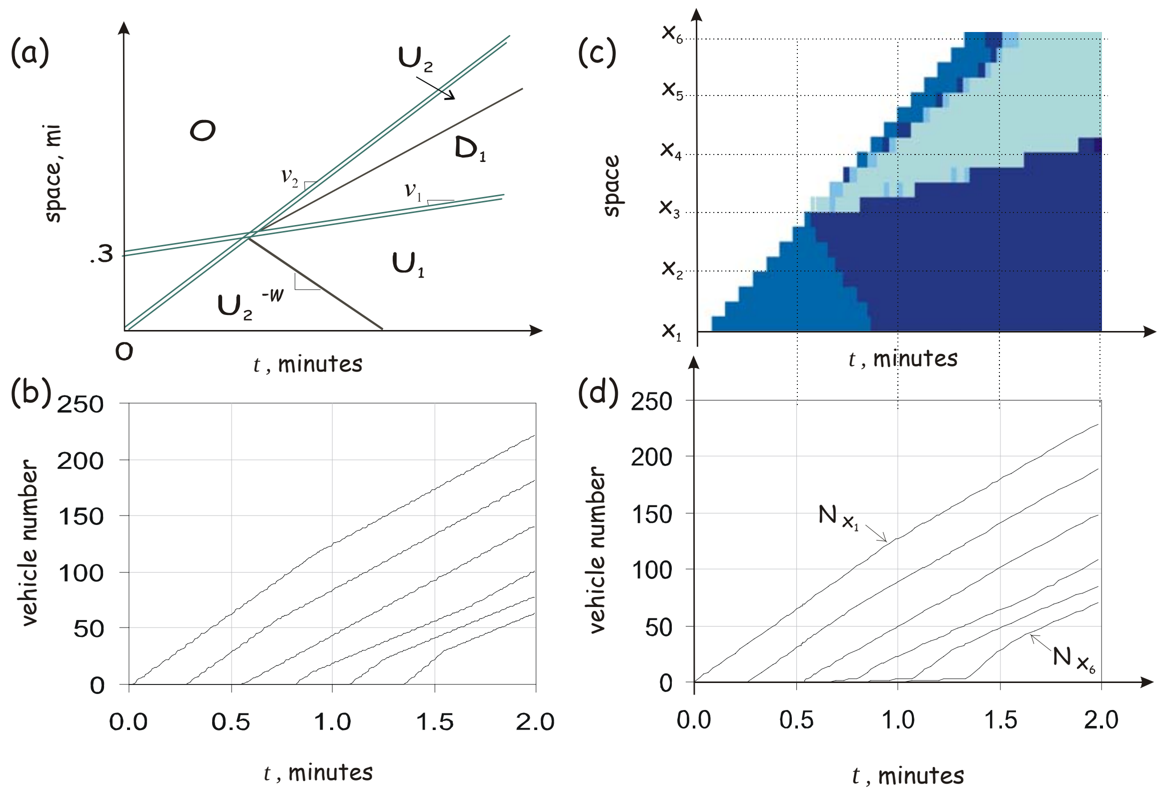

Figure 4: Comparison with example 2 in Daganzo and Laval 2003: (a) exact solution in the time-space diagram; (b) numerical *N*-curves with the method in Daganzo and Laval 2003 (Δt = 3 sec); (c) density map of the proposed method; (d) numerical *N*-curves with the proposed method (Δt = 3 sec).

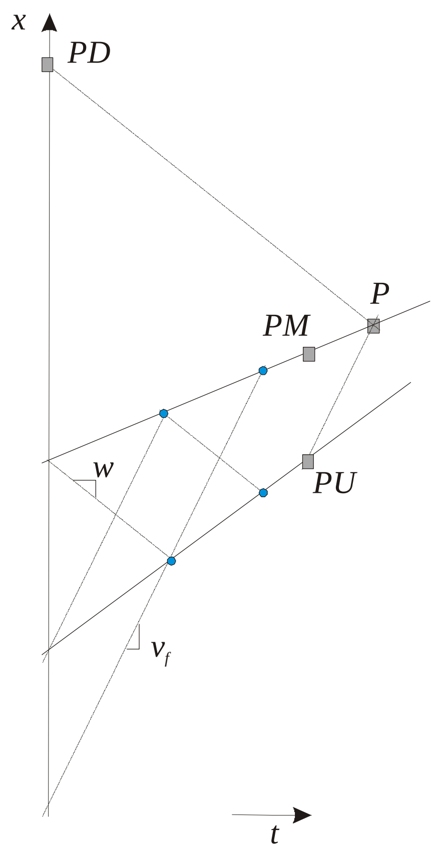

Figure A1: Basic properties of CRP's: dots denote possible singularities; squares denote possible anchors for point P.

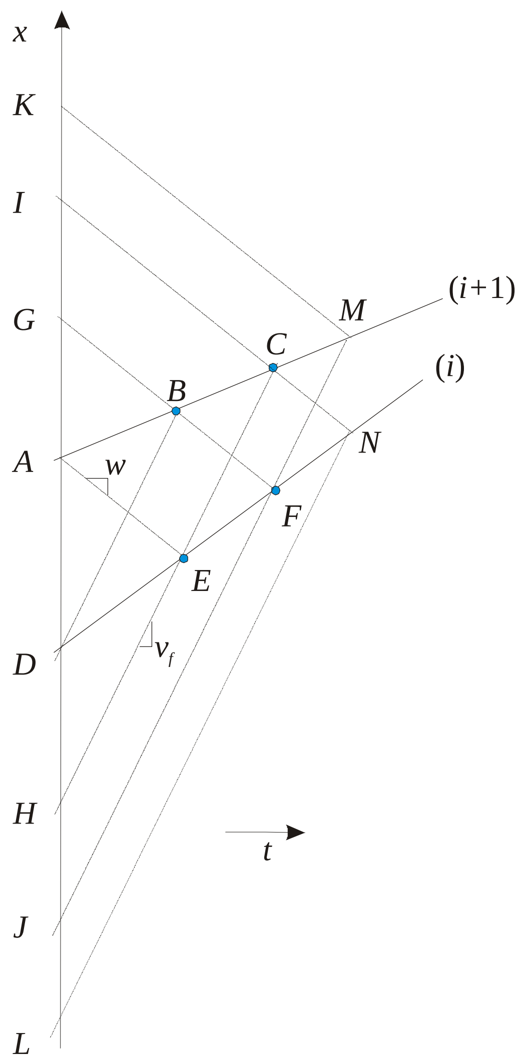

Figure A2: Points used in the solution of a CRP. (All the slanted lines that are not bottleneck trajectories have slopes - w or v_f .) Points B, E, C and F are potentially singular.