A multiclass car-following rule based on the LWR model

Ludovic Leclercq[1] and Jorge A. Laval[2]

[1] Laboratoire Ingénierie Circulation Transports - ENTPE / INRETS - Université de Lyon, Vaulx-en-Velin, France leclercq@entpe.fr
[2] Georgia Institute of Technology, Atlanta, USA jorge.laval@ce.gatech.edu

1 Introduction

The Lighthill-Whitham and Richards (LWR) model ([14], [17]) is well known for its simplicity, parsimony and its robustness to replicate basic traffic features. However, it considers traffic as a homogeneous flow. This can be a serious limitation when the traffic stream is composed of radically different vehicle classes, such as cars and heavy trucks near an uphill grade.
Extensive research has been conducted to introduce heterogeneity in the LWR model; e.g. [18], [19], [20], [3], [1], [15], [2]. All these extensions are based on the same principle: disaggregating the heterogeneous traffic flow into homogeneous and continuum vehicle classes that obey a conservation law with a specific fundamental diagram (FD). These models are solved numerically in Eulerian coordinates with methods such as the Godunov scheme [9]. However, these schemes are known to be very diffusive for hyperbolic systems of conservation laws [2].
Recent developments in traffic flow theory have led to efficient numerical schemes for solving the LWR model. They are derived in Lagrangian framework rather than in traditional Eulerian one; see for example [16], [4], [12], [13]. Additionally, variational theory [6], [7], [8] and its extension in Lagrangian coordinates [13] make it possible to prove that these schemes are exact for the LWR model when the FD is triangular. This is an important leap forward since current methods introduce numerical errors that can be devastating in practice. The aim of this paper is to extend the framework in [13] in order to incorporate multiple vehicle types, each one with a different car-following rule. In this way, the free-flow speed, the jam density and the wave-speed can be defined for each individual vehicle class. Note that the one-class car-following rule has already been coupled with a lane-changing model [10] and thus the proposed extension is fully compatible with the latter.
The sketch of the paper is as follows: section 1 will recall the Lagrangian formulation of the LWR model and its numerical resolution using (i) the Godunov scheme and (ii) the variational theory. Section 2 will introduce the

proposed extension, the resulting numerical schemes using (i) and (ii) above, and the conditions for the exactness of the numerical schemes. Finally, section 3 will present some numerical results focusing on the representation of a bimodal flow of trucks and cars.

2 The homogeneous LWR model in Lagrangian coordinates

Traditionally, the LWR model has been formulated in (x,t) coordinates. Introducing the cumulative count function $N(x,t)$, which represents the number of vehicles that have crossed position x by time t, allows to define a new coordinate system (N,t). These Lagrangian coordinates are fixed to a given fluid particle and move with it in space-time. The purpose is no longer to determine the local density $k(x,t)$ but the position $X(n,t)$ of vehicle n. In the remainder of the paper, capital N (respectively X) will stand for the $N(x,t)$ (respectively $X(n,t)$) function while n (respectively x) will define a specific value taken by this function. This section briefly presents the Lagrangian formulation of the LWR model. Details can be found in [13] or in [5].

2.1 Continuum formulation

Lagrangian conservation law

The LWR model in (n,t) coordinates can be expressed as a conservation law:

$$\partial_t s + \partial_n v = 0 \tag{1}$$

This model is fully described by the spacing s which corresponds to $1/k$. The speed v can be derived from the FD $v=V(s)$. Therefore, the LWR model can be described by the following hyperbolic equation in s:

$$\partial_t s + \partial_n V(s) = 0 \tag{2}$$

Lagrangian variational principle

The reader should refer to [6] and [7] for a complete description of variational theory in Eulerian coordinates. Indeed, the transformation to Lagrangian coordinates preserves the nature of the problem: the LWR model can be expressed as the Hamilton-Jacobi equation 3 derived from the FD.

$$\partial_t X = V(-\partial_n X) \tag{3}$$

All the results proven in [6] and [7] can thus be applied to the Lagrangian variational formulation of the LWR model. Notably, the value of X at a point P in the (n,t) plane, X_P, can be expressed as a least-cost path problem:

$$X_P = \min(B_\wp + C(\wp) : \forall \wp \in \mathbf{V} \cap \mathbf{P}_P), \text{ where} \\ \left| \begin{array}{l} \mathbf{V}\text{: set of all valid paths} \\ \mathbf{P}_P\text{: set of all paths from boundary condition to } P \\ B_\wp\text{: } X \text{ value at the beginning of the path} \\ C(\wp)\text{: cost of path } \wp \end{array} \right. \tag{4}$$

Analogously to [6], "waves" in (n,t) coordinates are characteristics where s is constant. They have slopes $u = \partial_s V(s)$ representing a passing rate. We define two types of passing rates: (a) u is a "possible passing rate" if there exists s such that $u = \partial_s V(s)$; (b) $\hat{u}$ is an "allowable passing rate" if $\min \partial_s V(s) \leq \hat{u} \leq \max \partial_s V(s)$. "Valid paths" are continuous and piecewise differentiable paths $n(t)$ in the (n,t) plane whose slopes $n'(t)$ are allowable passing rates. "Wave paths" are valid paths whose slopes are possible passing rates and are thus composed of a succession of waves.
The cost rate r on a wave path is given by $\mathrm{d}_t X$. The scalar r represents the speed of the Eulerian characteristic associated to the passing rate u.

$$r = \mathrm{d}_t X = \partial_t X + \partial_n X \partial_t n = v - su \tag{5}$$

As 3 holds and V is concave, one can express r only as a function $R(u)$ using the Legendre transformation as in [6]:

$$r = R(u) = \sup_s \{V(s) - su\} \tag{6}$$

The cost on a Lagrangian valid path P from B to P is thus:

$$C(\wp) = \int_{t_B}^{t_P} R(n'(t))dt \tag{7}$$

In the next section, we will show how the Lagrangian variational principle makes it possible to construct a numerical scheme which is exact under few restrictive assumptions.

2.2 Numerical resolution

Godunov scheme

In the Godunov scheme, the N-function is discretized in cells $i = 1, 2, \dots$ of size Δn and the spacing s is approximated by a constant value, s_i^t, which is updated at every time step Δt; see Figure 1. Since the flux function V in 2 is non-decreasing in s, the characteristic speed is always non-negative (traffic anisotropy). The Godunov method reduces then to the upwind method:

$$s_i^{t+\Delta t} = s_i^t + \frac{\Delta t}{\Delta n} \left(V(s_i^t) - V(s_{i-1}^t) \right) \tag{8}$$

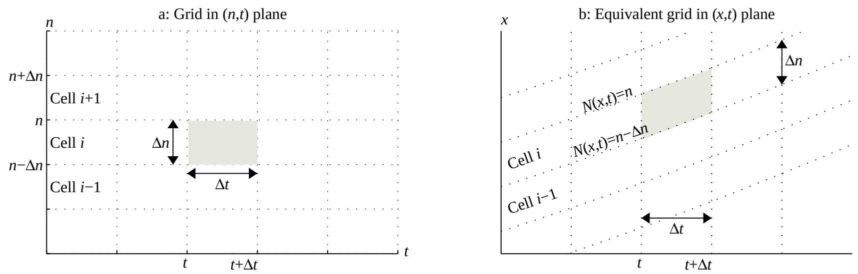

Fig. 1. Lagrangian grid

The Courant-Friedrich-Lewy's (CFL) condition 9 should hold to guaranty the stability and the convergence of 8.

$$\Delta n \geq \max_{s} |\partial_s V(s)| \, \Delta t \tag{9}$$

The Lagrangian Godunov scheme can also be expressed in terms of $X(n,t)$ by noting that the flux $V(s_i^t)$ at a boundary n of a cell i is:

$$\frac{X(n,t+\Delta t) - X(n,t)}{\Delta t} = V(s_i^t) = V\left(\frac{X(n-\Delta n,t) - X(n,t)}{\Delta n}\right) \tag{10}$$

Let us suppose now that the FD is triangular when expressed in terms of flow with respect to density:

$$V(s) = \min(v_m, w(\kappa s - 1)) \tag{11}$$

where v_m is the free-flow speed, w, the wave speed and κ the jam density; see Figure 2b. After simplification 10 becomes:

$$X(n,t+\Delta t) = \min(X(n,t)+v_m\Delta t, (1-\alpha)X(n,t)+\alpha X(n-\Delta n,t)-w\Delta t) \tag{12}$$

where $\alpha = w\kappa\Delta t/\Delta n$. If the CFL condition 9 is satisfied as an equality, $\alpha = 1$ and the scheme produces exact results [13]. In that case, one should fix either the time step Δt or the quantity of vehicle inside a Lagrangian cell Δn. For practical reason, it is easier to set Δn and to deduce $\Delta t = \Delta n/w\kappa$. Equation 12 reduces then to:

$$X\left(n, t+\frac{\Delta n}{w\kappa}\right) = min\left(X(n,t) + v_m\frac{\Delta n}{w\kappa}, X(n-\Delta n,t) - \frac{\Delta n}{\kappa}\right) \tag{13}$$

Note that when n is an integer and $\Delta n = 1$ then $X(n,t)$ corresponds to the position x_n^t of vehicle n at time t and $X(n\text{-}1,t)$ to the position x_{n-1}^t of its leader at the same time.

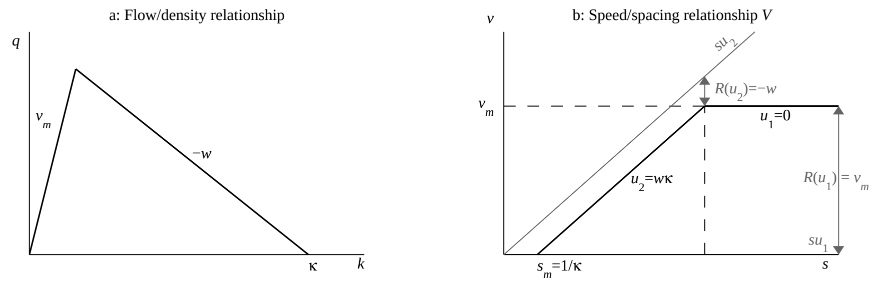

Fig. 2. Triangular fundamental diagram

Variational principle

Daganzo [7] proposed efficient methods to solve the LWR model using the concept of "sufficient networks". A "network" is defined as a directed graph where arcs are valid paths. A network is "sufficient" when the least-cost path through the network between every pair of nodes is optimum according to the continuous formulation of the model. In a sufficient network, the solution is exact at every node provided that the initial data is linear between two consecutive initial nodes. Otherwise, numerical errors are introduced as all optimum paths are not necessarily included in the network.
In Lagrangian coordinates, a sufficient network may easily be constructed when the FD is triangular. In this case, waves have only two possible velocities: $u_1 = 0$ (free-flow wave) and $u_2 = w\kappa$ (congestion wave). The resulting cost rates 6 are: $R(u_1) = v_m$ and $R(u_2) = -w$; see Figure 2b. Any geometric network formed by two families of parallel equidistant lines with slopes u_1 and u_2 and separated by Δn_1 and Δn_2 is sufficient; see Figure 3a. Therefore, with appropriate initial data, the solution at nodes is exact.
Since $u_1 = 0$ nodes are always lined-up along rows where n values are constant. Furthermore, if one sets $\Delta n_1 = \Delta n_2 = \Delta n$ nodes also line-up along "time-columns"; see Figure 3b. This defines a rectangular lattice in the (n,t) plane with $\Delta t = \Delta n/(w\kappa)$, which is very practical for computational implementation. Furthermore, with only two incoming arcs per node, the computation of 4 at each node is straightforward; i.e.:

$$\begin{aligned} X(n, t + \Delta t) &= \min\left(X(n,t) + \Delta_{(n,t)\to(n,t+\Delta t)}, X(n - \Delta n, t) + C_{(n-\Delta n,t)\to(n,t+\Delta t)}\right) \\ &= \min\left(X(n,t) + v_m \Delta t, X(n - \Delta n, t) - w\Delta t\right) \end{aligned} \tag{14}$$

where $C_{(n,t)\to(n',t')}$ is the cost of the arc between (n,t) and (n',t').

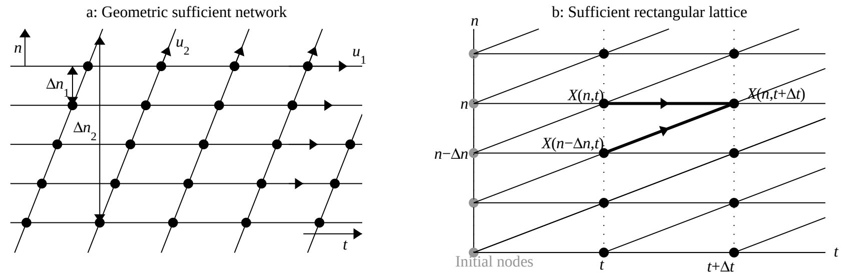

Fig. 3. Geometric networks associated to the Lagrangian variational principle

Notice that 14 and 13 are identical. Therefore, the Godunov and the variational schemes are equivalent when the FD is triangular and the CFL condition 9 is satisfied as an equality; i.e., when $\Delta t = \Delta n/(w\kappa)$.

3 Introducing different vehicle characteristics

3.1 Principle

The only way to distinguish different vehicle characteristics in Eulerian coordinates is to separate the global flow into several continuous homogeneous classes. The Lagrangian coordinate system makes this easier. Indeed, it allows for tracking vehicles and therefore applying a specific FD to them.
In the sequel, we suppose that m vehicle classes flow in proportions r_j, $j = 0, 1, \dots m-1$. Each class j is characterised by a triangular FD V_j with the following parameters: $v_{m,j}$, the free-flow speed, κ_j, the jam density and w_j the wave-speed. The free-flow speed $v_{m,j}$ depends both on vehicles' mechanical performances and legislation. The jam-density κ_j can be estimated as the inverse of the mean distance between two stopped vehicles of the same class j. This distance, l_j, is roughly equal to the mean vehicle length plus some buffer distance between vehicles (about one meter). The wave-speed w_j can be estimated as the ratio between l_j and the mean time between two successive departures inside a queue formed upstream from a traffic signal. Indeed, the wave-speed corresponds to the velocity of the starting wave observed when a traffic signal turns green. Note that the index $j = 0$ will always represent the class in which the mean vehicle length is minimum. Typically, it corresponds to passenger cars. For this latter class, the subscript j will be omitted.
For simplicity, we assume that the behavior of a given vehicle class is defined by the FD of its own class. We will see in the next section how the car-following rule is modified to account for different FDs.

3.2 Numerical resolution

The Lagrangian cell size Δn will now be set to 1, so that cells contain only one vehicle characterized by its specific FD. We can therefore drop the index i and refer to cells by the vehicle number, n. Numerical schemes will be derived either from the Godunov method or the variational principle.

Godunov scheme

Lebacque in [11] has demonstrated that the Godunov scheme can be applied with different FDs in each Eulerian cell. This also holds in Lagrangian coordinates. Thus, if the vehicle in cell n belongs to class j, 12 becomes:

$$X(n,t+\Delta t) = \min\left(X(n,t) + v_{m,j}\Delta t, (1-\alpha_j)X(n,t) + \alpha_j X(n-1,t) - w_j \Delta t\right) \tag{15}$$

where $\alpha_j = w_j \kappa_j \Delta t$. The CFL condition imposes that:

$$\forall k \in [0, m-1] \quad \Delta t \leq \frac{1}{w_j \kappa_j} \tag{16}$$

This condition cannot be satisfied as an equality for all classes. Thus, the numerical scheme is no longer exact. Numerical viscosity appears except for the class where the equality holds.

3.3 Variational principle

Considering a specific diagram for each vehicle does not modify the slope of free-flow waves in the Lagrangian network. Indeed, this slope $u_{1,j}$ is always equal to 0. Only the cost depends on vehicle class: $R(u_{1,j}) = v_{m,j}$. This can be easily accounted for as it does not change the structure of the network.
The slope of congestion wave, however, which depends on vehicle class: $u_{2,j} = w_j \kappa_j$. Cost on congestion waves is $R(u_{1,j}) = -w_j$. Thus it is no longer possible to build a sufficient network in the general case. To reduce numerical errors, one should set $\Delta t = 1/(w\kappa)$. Passenger car positions will then be exactly calculated. Congestion waves for all other classes will reach position $X_c(t)$ at time t; see Figure 4a. The location $X_c(t)$ is bounded by $X(n,t)$ and $X(n-1,t)$, and should be estimated as this point does not belong to the network. This will introduce numerical errors. If we suppose that the spacing is uniform in each Lagrangian cell, $X_c(t)$ can be deduced by:

$$X_c(t) = (1-\alpha_j)X(n,t) + \alpha_j X(n-1,t) \tag{17}$$

with $\alpha_j = w_j \kappa_j \Delta t$. The variational scheme is then:

$$X(n, t+\Delta t) = \min\left(X(n,t) + v_{m,j}\Delta t, X_c(t) - w_j \Delta t\right) \tag{18}$$

Note that 18 is fully equivalent to the Godunov scheme 15. This explains why this latter scheme is no longer exact for heterogeneous flow.

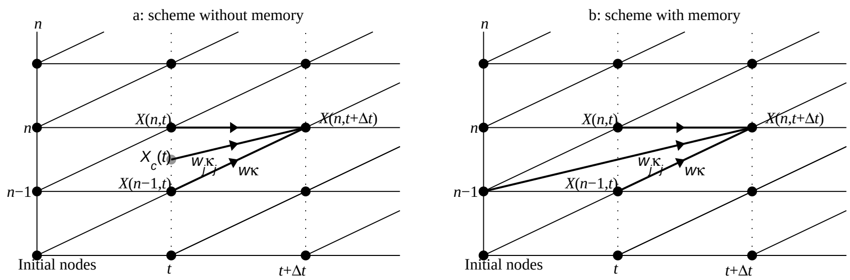

Fig. 4. Networks to account for different vehicle characteristics

Under some specific assumptions, it is still possible to have an exact numerical scheme. These assumptions are:

- the wave speed is the same for each j: $w_j = w$;
- the ratio l_j (mean vehicle length) over l (mean passenger car length) is an integer for each j.

Under these assumptions, the ratio $w\kappa/(w_j\kappa_j)$ is always an integer. Thus, each congestion wave can join a network node among the previous time steps; see Figure 4b. One has just to store the X values at the T previous time steps with $T = \max(w\kappa/(w_j\kappa_j))$. The numerical scheme becomes:

$$X(n,t{+}\Delta t) = \min\left(X(n,t) + v_{m,j}\Delta t, X\left(n-1, t-\left(\frac{\kappa}{\kappa_j}-1\right)\Delta t\right) - w_j\frac{\kappa}{\kappa_j}\Delta t\right) \tag{19}$$

Note that the above assumptions are not too restrictive. Indeed, many authors assume that vehicle's behaviours are almost the same in congestion [19], [15], [2]. Therefore a constant wave-speed for all classes is a reasonable assumption.

4 Numerical example

We are now going to focus on the bimodal case: light trucks and passengers cars. The FD parameters for both classes are:

- passenger cars: v_m =20 m/s; w =5 m/s; κ =0.2 veh/m (l =5 m);
- light trucks: $v_m, 1$ =12 m/s; w_1 =5 m/s; κ_1 =0.1 veh/m (l_1 =10 m).

Figure 5 presents the simulation results for a one-lane arterial road with a traffic signal located 400 m after the entrance. The cycle time is 90 s with 60 s of green time. The incoming flow is equal to 1080 veh/h. The ratio of light trucks r_1 is equal to 10% during 150 s and then switches to 40% until time t=340 s which corresponds to the end of the simulation.

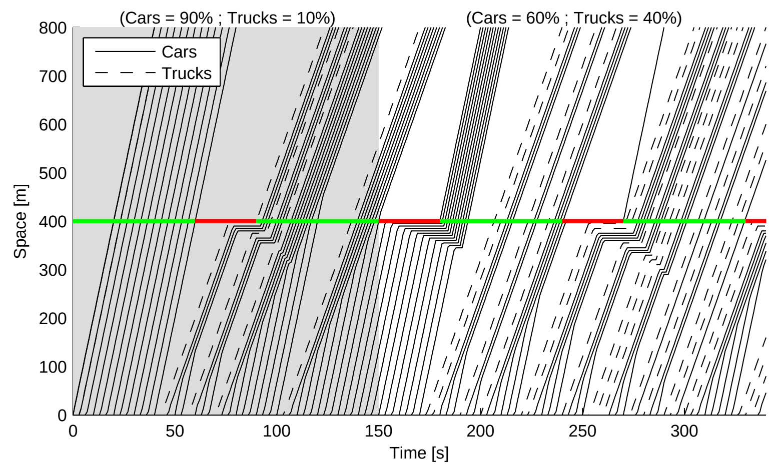

Fig. 5. Numerical example - exact variational scheme

The simulation results show that the proposed numerical scheme reproduces the different vehicle behaviour as expected. Notice how shockwaves propagate upstream as sharp discontinuities in vehicle speeds, regardless of the vehicle class it crosses. This appealing property is a consequence of the numerical method being exact for both vehicle classes.

5 Conclusion

This paper proposes an original method to account for a traffic heterogeneity in the LWR model. Contrary to previous approaches, this extension is introduced in Lagrangian coordinates rather than in classical Eulerian ones, which enable distinguishing vehicle characteristics via class-specific FDs. Furthermore, numerical resolution can be addressed with simple and effective schemes. The main advantage is that these schemes are exact under few restrictive assumptions.

For simplicity, we made the assumption that the behavior of a given vehicle class is defined by the FD of its own class. In particular, this amounts to assuming that the class of the vehicle immediately downstream of a given vehicle will not affect its behavior. While this is certainly true in freely-flowing traffic,

in congested conditions this may or may not be the case. Empirical evidence should be used to discern this matter. Fortunately, it is straight forward to extend the proposed model to capture such correlations: all one needs to do is define the FD for each pair of vehicle types.
The car-following rules proposed in this paper can be straightforwardly coupled with the lane-changing model in [10], yielding a complete microscopic multilane and multiclass model. Further research is being undertaken to validate this complete model with empirical data.

6 Acknowledgment

The authors are very grateful to Estelle Chevalier for her interesting comments and discussions and for helping to improve the final form of the paper.

References

1. Benzoni-Gavage S, Colombo R.M (2003) European Journal on Applied Mathematics 14:587–612
2. Chanut S (2005) First-order macroscopic model for mixed traffic including moving bottleneck effects. In: Mahmassani H.S (eds) 16th ISTTT. Elsevier, London
3. Chanut S, Buisson C (2003) TRR 1852:209–219
4. Daganzo C.F (2006) Transportation Research part B 40(5):396–403
5. Daganzo C.F (2006b) Networks and Heterogeneous Media 1(4):601–619
6. Daganzo C.F (2005) Transportation Research part B 39(2):187–196
7. Daganzo C.F (2005b) Transportation Research part B 39(10):934–950
8. Daganzo C.F, Menendez M (2005) A variational formulation of kinematic waves: bottlenecks properties and examples. In: Mahmassani H.S (eds) 16th ISTTT. Elsevier, London
9. Godunov S.K (1959) Mat. Sb. 47:271–290
10. Laval J.A, Leclercq L (2007) A microscopic theory of lane-changing. Submitted for publication. Available as Research Report LICIT 06-05, INRETS, Bron
11. Lebacque J.P (1996) The Godunov scheme and what it means for first order traffic flow models. In: Lesort J.B (eds) 13th ISTTT. Elsevier, London
12. Leclercq L (2007) Transportation Research part B 41(7):701–709
13. Leclercq L, Laval J.A, Chevallier E (2007) The Lagrangian coordinates and what it means for first order traffic flow models. In: Allsop R.E, Bell M.G.H, Heydecker B.G (eds) 17th ISTTT. Elsevier, London
14. Lighthill M.J, Whitham J.B (1955) On kinematic waves II: A theory of traffic flow in long crowded roads. Proceedings of the Royal Society, A229:317–345
15. Logghe S (2003) Dynamic modeling of heterogenous vehicular traffic. PhD Thesis, Kathlieke Universiteit Leuven, Leuven
16. Newell G.F (2002) Transportation Research part B 36(3):195–205
17. Richards P.I (1956) Operations Research, 4:42–51
18. Wong G.C.K, Wong S.C (2002) Transportation Research part A 36(9):827–841
19. Zhang H.M, Jin W.L (2002) TRR, 1802:197–204
20. Zhu Z, Chang G.L, Wu T (2003) TRR, 1852:201–208