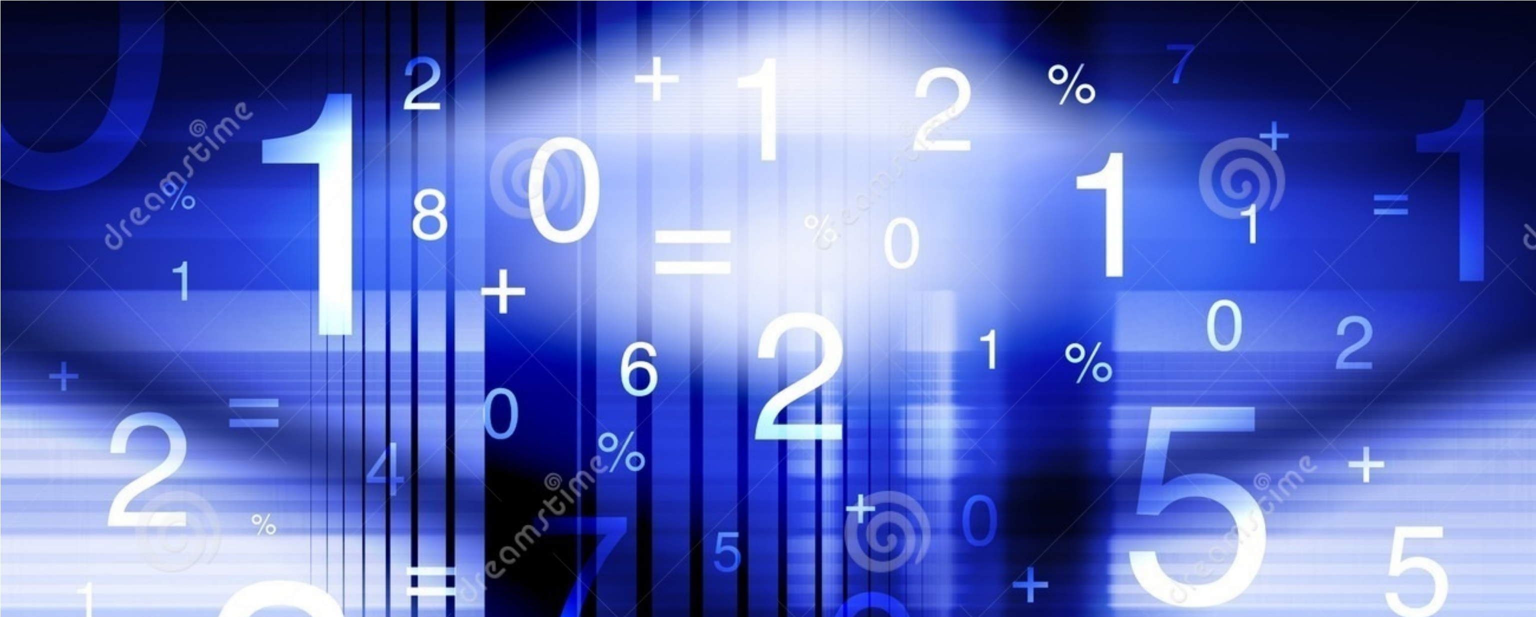
Notes on Probability and Statistics



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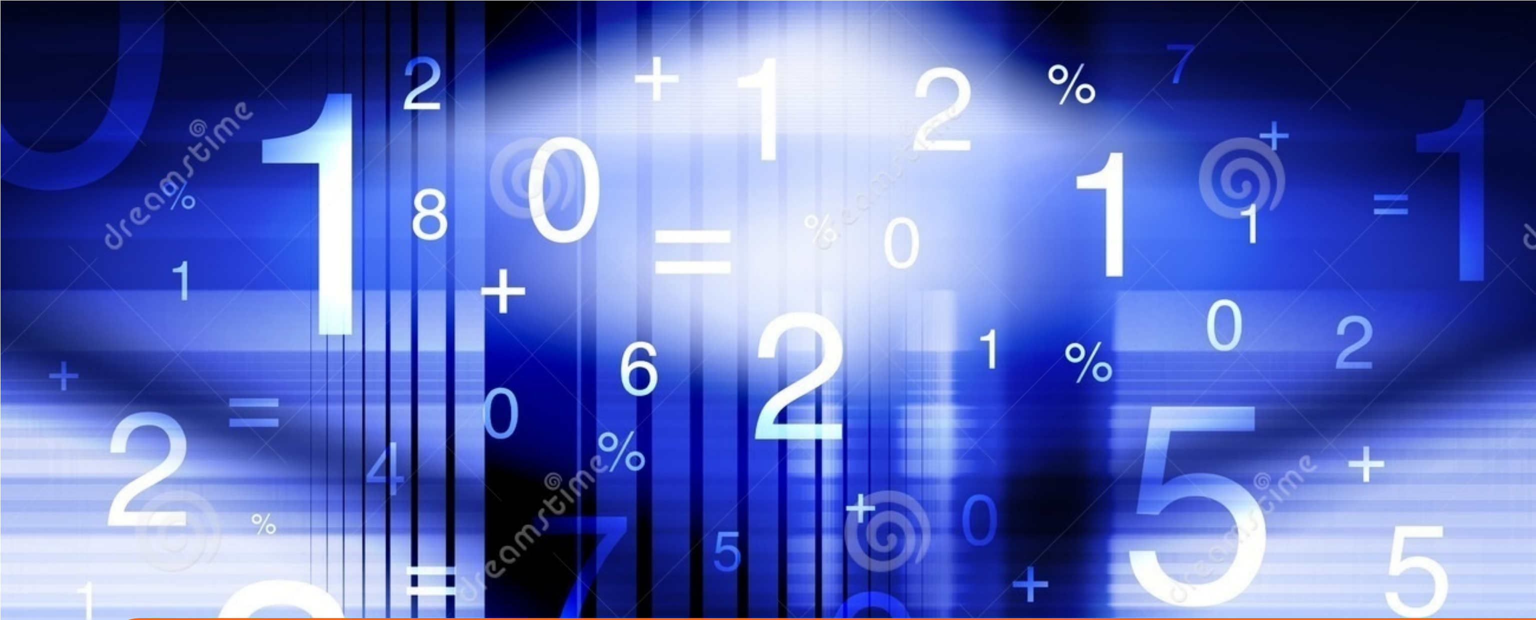
Probability



5. Function of Random Variables

5.1 One Random Viable

We are interested in the distribution of

$$\boxed{Y = g(X)} \tag{5.1}$$

when the distribution of X is known.

> **If we only need $\mathbf{E}(Y)$ and $V(Y)$.** It is not necessary to calculate the distribution of Y:
> $\mathrm{E}(Y) = E[g(X)] = \int g(x) f_X(x)\, dx$
> $V(Y) = E[g(X)^2] - [E[g(X)]^2$

STOP! $\mathrm{E}(Y) = E(g(X)) \neq g(E(X))$.

naïve approach: $g(E(X))$
correct approach: $E(g(X))$.

Example 1. X has the following distribution.

$$\begin{aligned} \mathrm{P}(X=-1) &= 0.2 \\ \mathrm{P}(X=0) &= 0.5 \\ \mathrm{P}(X=1) &= 0.3 \end{aligned}$$

Find the distribution of $Y = X^2$.

Solution: Let

$$Y = X^2$$

Using option one above, we have

$$\begin{aligned} \mathrm{P}(Y=0) &= \mathrm{P}(X^2=0) \\ &= \mathrm{P}(X=0) \\ &= 0.5 \\ \mathrm{P}(Y=1) &= \mathrm{P}(X^2=1) \\ &= \mathrm{P}(X=-1)+\mathrm{P}(X=1) \\ &= 0.5 \end{aligned}$$

$\square$

Example 2. The probability density function of X is given by the Uniform distribution in (0, 1):

$$f_X(x) = \begin{cases} 1 & ; \quad 0 \leq x \leq 1 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

Find the distribution of $Y = e^X$.

Solution: Let $Y = e^X$. Therefore,

$$\begin{aligned} \mathrm{F}_Y(y) &= \mathrm{P}(Y \leq y) = \mathrm{P}\left(e^X \leq y\right) = \mathrm{P}(X \leq \log y) = F_X(\log y) \\ &= \int\limits_{-\infty}^{\log y} f_X(x)\,\mathrm{d}x = \int\limits_{0}^{\log y} \mathrm{d}x = \log y \end{aligned}$$

Therefore, differentiating,

$$f_Y(y) = \frac{\mathrm{dF}_Y(y)}{\mathrm{d}y} = \frac{\mathrm{d}\log y}{\mathrm{d}y} = \frac{1}{y}$$

$\square$

5.1.1 Single Discrete Random Variable

If the function $g(X)$ is monotonic then, the recipe is

$$\boxed{p_Y(y) = \begin{cases} p_X(g^{-1}(y)) & \text{if } g^{-1}(y) \in S_X \\ 0 & \text{otherwise} \end{cases}} \tag{5.2}$$

Notice from the figure that:

$$S_Y = \{y_1, y_2 \ldots y_5\}$$

$$P(Y = y_i) = P(X = x_i), \quad i = 1, 2, \ldots 5$$

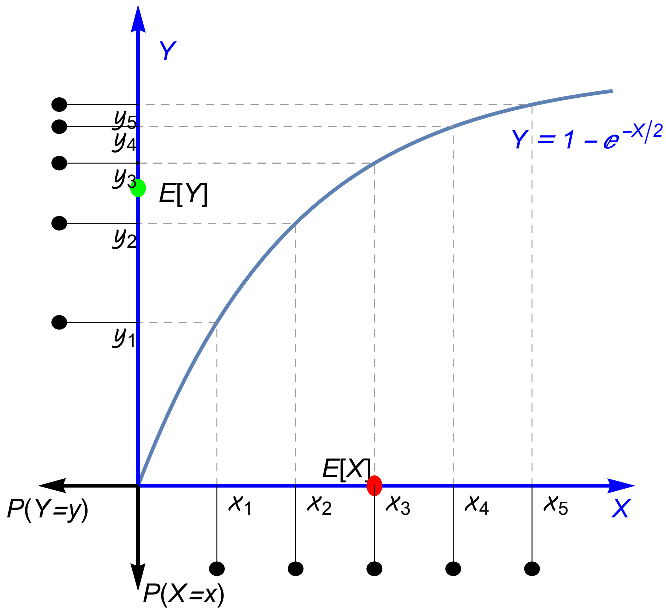
$$x_i = g^{-1}(y_i) = -2\log(1 - y_i)$$

$$P(Y = y_i) = P(X = g^{-1}(y_i))$$

or,

$$\boxed{p_Y(y_i) = p_X(g^{-1}(y_i))}$$

Notice: $E(g(X)) < g(E(X))$
(always true for concave functions)



From the figure for $g(X) = 1/X$:

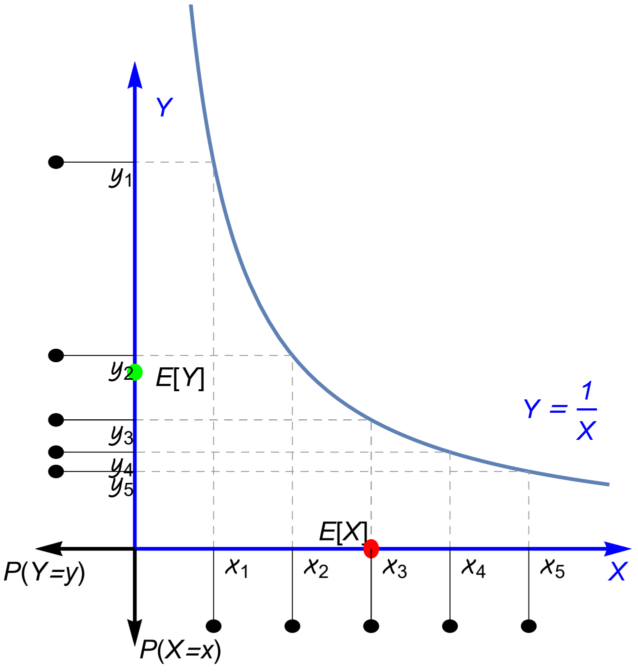
$$S_Y = \{y_1, y_2 \ldots y_5\}$$

$$P(Y = y_i) = P(X = x_i), \quad i = 1, 2, \ldots 5$$
$$x_i = g^{-1}(y_i) = 1/y_i$$
$$P(Y = y_i) = P(X = g^{-1}(y_i))$$

or,

$$p_Y(y_i) = p_X(g^{-1}(y_i))$$

Notice: $E(g(X)) > g(E(X))$
(always true for convex functions)



From the figure for $g(X) = X^3$:

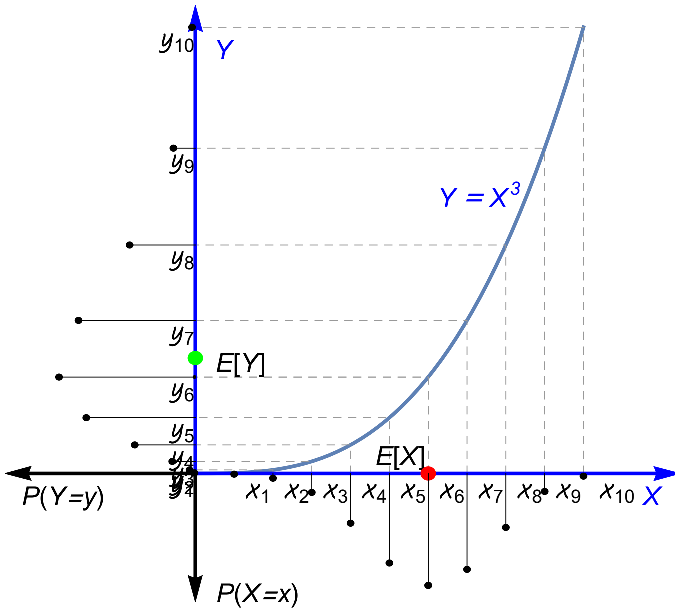
$$S_Y = \{y_1, y_2 \dots y_{10}\}$$

$$P(Y = y_i) = P(X = x_i), \quad i = 1, 2, \dots 10$$
$$x_i = g^{-1}(y_i)$$
$$P(Y = y_i) = P(X = g^{-1}(y_i))$$

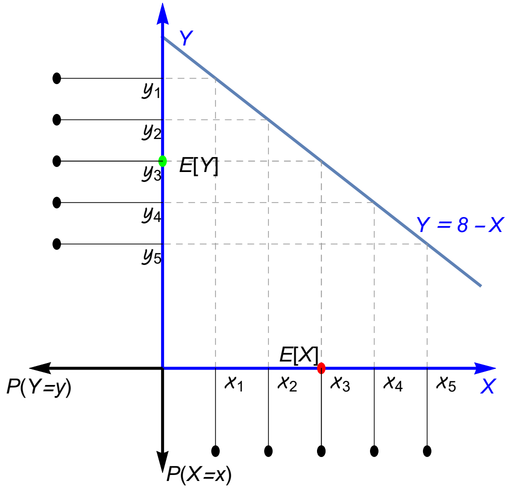
Therefore,

$$p_Y(y_i) = p_X(g^{-1}(y_i))$$

Notice: $E(g(X)) > g(E(X))$



When $g(x)$ is **linear**, the shape of the distribution remains the same:



For **non-monotonic functions**, suppose the solution of $y = g(x)$ has k roots: x_1^*, x_2^*, ..., x_k^*. Therefore,

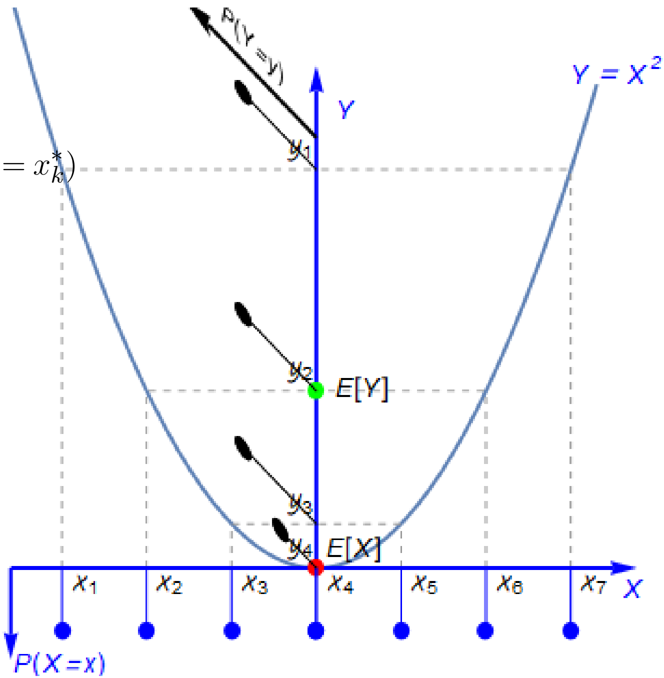
$$p_Y(y) = P(X = x_1^*) \cup P(X = x_2^*) \cup \ldots \cup P(X = x_k^*)$$
$$= \sum_{i=1}^{k} P(X = x_i^*)$$

In the figure, $x_1^* = \sqrt{y}$ and $x_2^* = -\sqrt{y}$.

$$S_Y = \{y_1, y_2 \ldots y_4\}$$

$$p_Y(y_i) = p_X(\sqrt{y_i}) + p_X(-\sqrt{y_i}), i = 1, 2, 3$$
$$p_Y(y_4) = p_X(0)$$

Notice: $E(g(X)) > g(E(X))$



From the above figures we have clarified the following important inequalities,

Jensen's inequalities :

If $g(X)$ is **convex** then:

$$E(g(X)) \geq g(E(X))$$

If $g(X)$ is **concave** then:

$$E(g(X)) \leq g(E(X))$$

If $g(X)$ is **linear** then:

$$E(g(X)) = g(E(X))$$

Example 3. Given $X \sim \mathrm{Bin}(n,p)$ and $Y = e^X$. What is the distribution of Y, $p_Y(y)$?

Solution: The recipe:

$$p_Y(y) = \begin{cases} p_X(g^{-1}(y)) & \text{if } g^{-1}(y) \in S_X \\ 0 & \text{otherwise} \end{cases}$$

Here, $x = g^{-1}(y) = \log y$ and $p_X(x) = \binom{n}{x} p^x (1-p)^{n-x}$, so:

$$p_Y(y) = \begin{cases} \binom{n}{\log y} (p)^{\log y} (1-p)^{n-\log y} & \text{when } \log y \text{ is an integer} \\ 0 & \text{otherwise} \end{cases}$$

Note: $S_Y = \{1, e, e^2, ..., e^n\}$

$\square$

Example 4. $X \sim \text{Bin}(n,p)$ and $Y = X^2$. What is the distribution of Y, $p_Y(y)$?

Solution: $x = g^{-1}(y) == \pm\sqrt{y}$, so $x_1^* = \sqrt{y}$ and $x_2^* = -\sqrt{y}$. In this case $x_2^* \notin S_X$ because it is negative, and therefore

$$p_Y(y) = \begin{cases} \binom{n}{\sqrt{y}}(p)^{\sqrt{y}}(1-p)^{n-\sqrt{y}} & \text{when } \sqrt{y} \text{ is an integer} \\ 0 & \text{otherwise} \end{cases}$$

□

5.1.2 Single Continuous Random Viable

Here $Y = g(X)$ is a monotone function and $f_X(x)$ is known. The PDF $f_Y(y)$ is

$$\boxed{f_Y(y) = f_X\left(g^{-1}(y)\right)\left|\frac{d}{dy}g^{-1}(y)\right|} \tag{5.3}$$

The derivation is analogous to the discrete case, where the key idea was $P(Y = y) = P(X = g^{-1}(y))$. In the continuous case this reads:

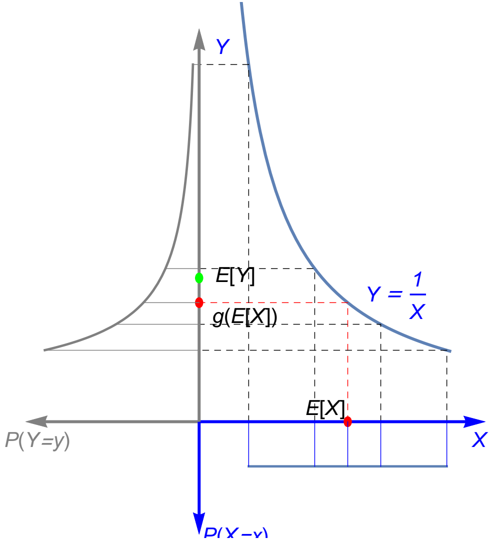
$$f_Y(y)\ dy = f_X\left(g^{-1}(y)\right)\ dx$$

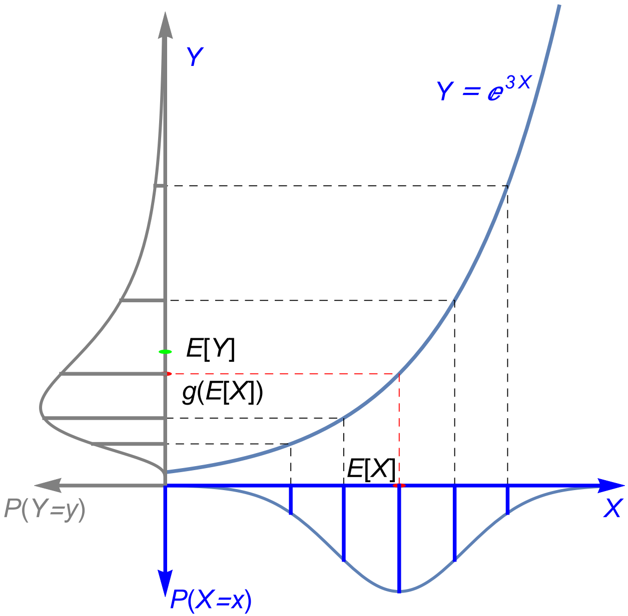
and from the figure seen in class

$$dx = \left|\frac{d}{dy}g^{-1}(y)\right| dy$$

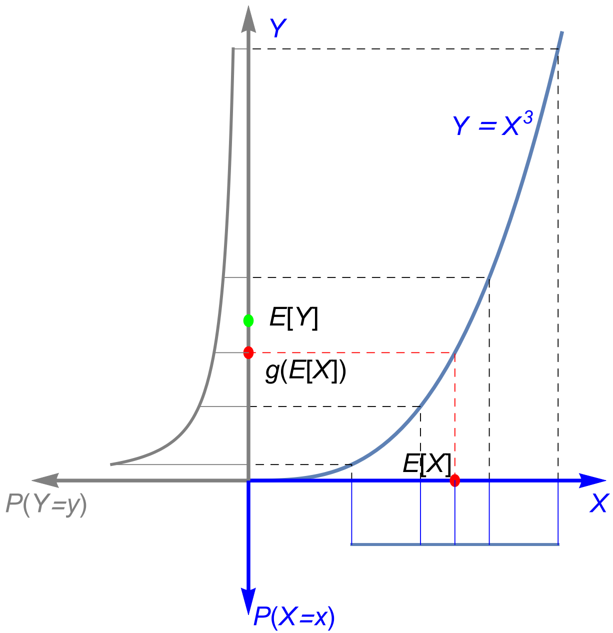
which establishes the result.

Notice how the shape of $f_Y(y)$ changes due to $Y = g(X)$:





Y
$Y = e^{3X}$
E[Y]
g(E[X])
E[X]
P(Y=y)
X
P(X=x)


Y
Y = X^3
E[Y]
g(E[X])
E[X]
X
P(Y=y)
P(X=x)

Example 5. $Y = e^X$, $X \sim$N(μ,σ^2). Find $f_Y(y)$.

Solution:

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2 \big/ 2\sigma^2}$$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dg^{-1}}{dy} \right|$$

Since $Y = e^X = g(x)$, $g^{-1}(y) = \log(y)$, and $\left|\frac{dg^{-1}}{dy}\right| = \frac{1}{y}$. Therefore,

$$f_Y(y) = \frac{1}{y\sigma\sqrt{2\pi}} e^{-(\log y-\mu)^2 \big/ 2\sigma^2}$$

which corresponds to the lognormal distribution, $Y \sim LN(\mu,\sigma^2)$. $\square$

Example 6. Let the random variable X be exponentially distributed with mean 2. You are interested in $Y = e^{-2X}$. Find $f_Y(y)$.

Example 7. The absolute velocity (X) of particles in a gas follows a Maxwell distribution, with the PDF

$$f_X(x) = \begin{cases} \frac{4x^2}{a^3\sqrt{\pi}} \exp(-\frac{x^2}{a^2}), & x > 0 \\ 0, & otherwise \end{cases}$$

where a is a constant. Determine the PDF $f_Y(y)$ for the particle kinetic energy $Y = \frac{1}{2}mX^2$, where m is the mass of a particle.

Solution: Answer:

$$\begin{aligned} g(X) &= \frac{1}{2}mX^2 \\ g^{-1}(y) &= \pm\sqrt{\frac{2y}{m}} \\ \left|\frac{dg^{-1}}{dy}\right| &= \frac{1}{\sqrt{2my}} \end{aligned}$$

$$
\begin{aligned}
f_Y(y) &= f_X\left(\sqrt{\frac{2y}{m}}\right)\frac{1}{\sqrt{2my}} + f_X\left(-\sqrt{\frac{2y}{m}}\right)\frac{1}{\sqrt{2my}} \\
&= f_X\left(\sqrt{\frac{2y}{m}}\right)\frac{1}{\sqrt{2my}} + 0 \\
&= \frac{8}{a^3\sqrt{\frac{2\pi m^3}{y}}} e^{-\frac{2y}{my^2}}, y > 0
\end{aligned}
$$

$\square$

5.2 Two Random Viables

Here

$$Z = g(X, Y)$$

When X, Y are discrete, assuming $p_{X,Y}$ is known:

$$p_Z(z) = \sum_{\text{all } (x,y):\ z=g(x,y)} p_{X,Y}(x, y)$$

If the function $g(X, Y)$ is monotone:

$$\begin{aligned} p_Z(z) &= \sum_{x \in S_X} p_{X,Y}(x, g^{-1}) \quad \text{with:} \quad g^{-1} = g^{-1}(x, z) \quad \text{or:} \\ &= \sum_{y \in S_Y} p_{X,Y}(g^{-1}, y) \quad \text{with:} \quad g^{-1} = g^{-1}(y, z) \end{aligned}$$

Example 8. Suppose Z=X+Y where X$\sim$ Poi(λ), Y$\sim$ Poi(μ) and X and Y are independent. What is the $p_Z(z)$?

Solution: Given: $p_X(\text{x}) = \frac{\lambda^x}{x!}e^{-\lambda}$ and $p_Y(y) = \frac{\mu^y}{y!}e^{-\mu}$

$$
\begin{aligned}
p_Z(z) &= \sum_{\text{all } (x,y):\ z=x+y} p_{X,Y}(x,y) \\
&= \sum_{x \in S_X} p_{X,Y}(x, g^{-1}) \quad \text{with:} \quad g^{-1} = g^{-1}(x,z) = z - x \\
&= \sum_{x=0}^{z} p_X(x) \cdot p_Y(z-x) \\
&= \sum_{x=0}^{z} \frac{\lambda^x}{x!} \frac{\mu^{z-x}}{(z-x)!} e^{-(\lambda+\mu)} \\
&= e^{-(\lambda+\mu)} \sum_{x=0}^{z} \frac{\lambda^x \mu^{z-x}}{x!(z-x)!} \\
&= \frac{(\lambda+\mu)^z}{z!} e^{-(\lambda+\mu)} \\
&= \text{Poi}(\lambda+\mu)
\end{aligned}
$$

When X, Y are continuous, assuming f_X and f_Y are known:

$$f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}\left(g^{-1}, y\right) \left|\frac{\partial}{\partial z} g^{-1}\right| dy \quad \text{with:} \quad g^{-1} = g^{-1}(z, y)$$

Alternatively, we can also use

$$f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}\left(x, g^{-1}\right) \left|\frac{\partial}{\partial z} g^{-1}\right| dx \quad \text{with:} \quad g^{-1} = g^{-1}(x, z)$$

> **Example 9.** Suppose Z=X+Y where X∼Exp(λ), Y∼Exp(μ) and X and Y are independent. What is $f_Z(z)$?

Solution: Given: $f_X(x) = \lambda e^{-\lambda x}$ and $f_Y(y) = \mu e^{-\mu y}$
Since we know:

$$f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}(g^{-1}, y) \left| \frac{dg^{-1}}{dz} \right| dy \quad with \quad g^{-1} = g^{-1}(z, y)$$

Since X and Y are independent:

$f_{X,Y}(x, y) = f_X \cdot f_Y = \lambda\mu e^{-\lambda x + \mu y}$.

To obtain $\left|\frac{dg^{-1}}{dz}\right|$, we let $x = g^{-1} = z - y$. Therefore:

$$\left| \frac{dg^{-1}}{dz} \right| = \left| \frac{d}{dz}(z - y) \right| = |1| = 1$$

and

$$f_{X,Y}(g^{-1}, y) = \lambda\mu e^{-(\lambda(z-y)+\mu y)} dy$$

Therefore, $f_Z(z)$ can be calculated as (from the figure seen in class):

$$
\begin{aligned}
f_Z(z) &= \int_0^z f_{X,Y}(z-y,y)\left|\frac{dg^{-1}}{dz}\right| dy \\
&= \lambda\mu e^{-\lambda z}\int_0^z e^{-y(\mu-\lambda)}dy \\
&= \frac{\lambda\mu}{\mu-\lambda}(e^{-\lambda z}-e^{-\mu z})
\end{aligned}
$$

Note: If X and Y have the same rate, it would be a Gamma (Erlang) distribution. $\square$

Example 10. Suppose $Z = X \cdot Y$ where X~Exp(λ), Y~Exp(μ) and X and Y are independent. What is $f_Z(z)$?

Solution:

$$y = \frac{z}{x}, \left|\frac{dg^{-1}}{dz}\right| = \frac{1}{x}$$

$$f_Z(z) = \int_0^{\infty} \lambda\mu e^{-(\lambda x + \mu \frac{z}{x})} \frac{1}{x} dx$$

$\square$

Example 11. Suppose Z=X+Y where X$\sim$N(μ_X,σ_X), Y$\sim$N(μ_y,σ_y) and X and Y are independent. Show that:

$$Z \sim N(\mu_X + \mu_y, \sigma_X^2 + \sigma_y^2)$$

5.2.1 What if the distribution of X is unknown? (not covered)

If we only only know $\mathrm{E}(X)=\mu$ and $\mathrm{V}(X)=\sigma^2$, we can still approximate $\mathrm{E}(Y)$ and $\mathrm{V}(Y)$ by Taylor Series around the mean of X:

$$Y=g(X)\approx g(\mu)+(X-\mu)g'(\mu)+\frac{1}{2}(X-\mu)^2 g''(\mu)+\dots$$

2nd-order approximation for E[Y]:

$$\boxed{\mathrm{E}(Y)\approx g(\mu)+\frac{1}{2}g''(\mu)\sigma^2} \tag{5.4}$$

1st-order approximation for V[Y]:

$$\boxed{\mathrm{V}(Y)\approx g'(\mu)^2\sigma^2} \tag{5.5}$$

Several random variables

Let $Y = g(X_1, X_2, \ldots X_n)$ and recall the vector notation:

$$\mathbf{X} = (X_1, X_2, \ldots X_n)^T$$

The joint PMF is not known; all we know are:

$$\mathrm{E}(X_i) = \mu_i \qquad , \qquad \mathrm{V}\left(X_i\right) = \sigma_i^2 \qquad , \qquad \mathrm{Cov}(X_i, X_j) = \sigma_{ij}$$

and $\boldsymbol{\mu} = \{\mu_1, \mu_2, \ldots \mu_n\}$. A second-order Taylor series expansion of the scalar-valued function $g(\cdot)$ can be written compactly as

$$g(\mathbf{X}) = g(\boldsymbol{\mu}) + (\mathbf{X} - \boldsymbol{\mu})^\mathsf{T}\mathbf{G} + \frac{1}{2!}(\mathbf{X} - \boldsymbol{\mu})^\mathsf{T}\mathbf{H}(\mathbf{X} - \boldsymbol{\mu}) + \cdots$$

where $\mathbf{G}$ and $\mathbf{H}$ are the gradient vector and the Hessian matrix of g evaluated at $\mathbf{X}) = \boldsymbol{\mu}$, resp.

2nd-order approximation for $\mathrm{E}(Y)$:

$$\boxed{\begin{aligned}
\mathrm{E}(Y) &\approx g(\boldsymbol{\mu}) + \frac{1}{2}\mathbf{e}^T(\Sigma_{\mathbf{X}} \odot \mathbf{H})\mathbf{e} \\
&= g(\boldsymbol{\mu}) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij} \cdot \left(\frac{\partial^2 g}{\partial X_i \partial X_j}\right) \\
&= g(\boldsymbol{\mu}) + \frac{1}{2}\sum_{i=1}^{n}\sigma_i^2 \cdot \left(\frac{\partial^2 g}{\partial X_i^2}\right) + \underbrace{\sum_{i=1}^{n}\sum_{j=i+1}^{n}\sigma_{ij} \cdot \left(\frac{\partial^2 g}{\partial X_i \partial X_j}\right)}_{0 \text{ if } X_i\text{'s are independent}}
\end{aligned}} \tag{5.6}$$

where $\mathbf{e}$ is the column vector whose entries are all 1's, and $\odot$ is the Hadamard product, which takes two matrices of the same dimensions and produces another matrix where each element i, j is the product of elements i, j of the original two matrices. It should not be confused with the more common matrix product.

1st-order approximation for $\mathrm{V}(Y)$:

$$\boxed{\begin{aligned} \mathrm{V}(Y) &\approx \mathbf{G}^T \Sigma_{\mathbf{X}} \mathbf{G} \\ &= \sum_{i=1}^{n} \sigma_i^2 \cdot \left(\frac{\partial g}{\partial X_i} \right)^2 \end{aligned}} \tag{5.7}$$

Example 12. — * The hydraulic head loss in a pipe may be determined by the Darcy-Weisbach equation as follows:

$$H = \frac{fLV^2}{2Dg}$$

where:
L=length of a pipe, V=flow velocity of water in a pipe, D=pipe diameter, f=coefficient of friction, g=gravitational acceleration=32.2 ft/sec^2. Suppose a pipe has the following properties:

i	X_i	μ_i	δ_i
1	L	100.	0.1
2	D	1.	0.1
3	f	0.02	0.2
4	V	10.	0.15

(a) Approximate the mean and standard deviation of the hydraulic head loss of the pipe.

Solution: (a) We have:

	X_i	μ_i	σ_i
1	L	100.	10.
2	D	1.	0.1
3	f	0.02	0.004
4	V	10.	1.5

$$\mathbf{G} = \begin{pmatrix} \frac{fV^2}{2Dg} \\ -\frac{fLV^2}{2D^2g} \\ \frac{LV^2}{2Dg} \\ \frac{fLV}{Dg} \end{pmatrix} \text{ and } \mathbf{H} = \begin{pmatrix} 0 & -\frac{fV^2}{2D^2g} & \frac{V^2}{2Dg} & \frac{fV}{Dg} \\ -\frac{fV^2}{2D^2g} & \frac{fLV^2}{D^3g} & -\frac{LV^2}{2D^2g} & -\frac{fLV}{D^2g} \\ \frac{V^2}{2Dg} & -\frac{LV^2}{2D^2g} & 0 & \frac{LV}{Dg} \\ \frac{fV}{Dg} & -\frac{fLV}{D^2g} & \frac{LV}{Dg} & \frac{fL}{Dg} \end{pmatrix}, \text{ evaluating at } \boldsymbol{\mu} \text{ gives:}$$

$$\mathbf{G} = \begin{pmatrix} 0.031 \\ -3.106 \\ 155.28 \\ 0.621 \end{pmatrix} \text{ and } \mathbf{H} = \begin{pmatrix} 0. & -0.03 & 1.55 & 0.01 \\ -0.03 & 6.21 & -155.28 & -0.62 \\ 1.55 & -155.28 & 0. & 31.06 \\ 0.01 & -0.62 & 31.06 & 0.06 \end{pmatrix}$$

and the mean and variance of H are approximately:

$$
\begin{aligned}
\mathrm{E}(H) &= g(\boldsymbol{\mu}) + \frac{1}{2}\sum_{i=1}^{n}\sigma_i^2 \cdot \left(\frac{\partial^2 g}{\partial X_i^2}\right) \\
&= \frac{0.02 \times 100 \times 10^2}{2 \times 1 \times 32.2} + \frac{1}{2}(6.21 \times 0.01 + 0.06 \times 2.25) \\
&= 3.20652
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{V}\left(H\right) &= \sum_{i=1}^{n}\sigma_i^2 \cdot \left(\frac{\partial g}{\partial X_i}\right)^2 \\
&= 0.000961 \times 100. + 9.64724 \times 0.01 + 24111.9 \times 0.000016 + 0.385641 \times 2.25 \\
&= 1.44605
\end{aligned}
$$

$\square$

Example 13. Refer to example 12 and assume that the correlation between D and f is 0.7 and between V and f, 0.4. Show that the expected value and variance of H are now 3.237 and 1.639, respectively.

Solution: Hint:

$$
\begin{aligned}
\mathrm{E}(H) = {} & \frac{0.02 \times 100 \times 10^2}{2 \times 1 \times 32.2} + \frac{1}{2}(-155.28\sigma_{2,3} - 155.28\sigma_{3,2} + 31.056\sigma_{3,4} + 31.056\sigma_{4,3} + \\
& + 6.211\sigma_2^2 + 0.062\sigma_4^2) \\
\mathrm{V}(H) = {} & -3.106\left(155.28\sigma_{3,2} - 3.106\sigma_2^2\right) + 0.621\left(155.28\sigma_{3,4} + 0.621\sigma_4^2\right) + \\
& + 155.28\left(-3.106\sigma_{2,3} + 0.621\sigma_{4,3} + 155.28\sigma_3^2\right) + 0.000961\sigma_1^2
\end{aligned}
$$

□

5.3 Important distributions for statistics

5.3.1 The chi-square distribution with r degrees of freedom

The density function is given by:

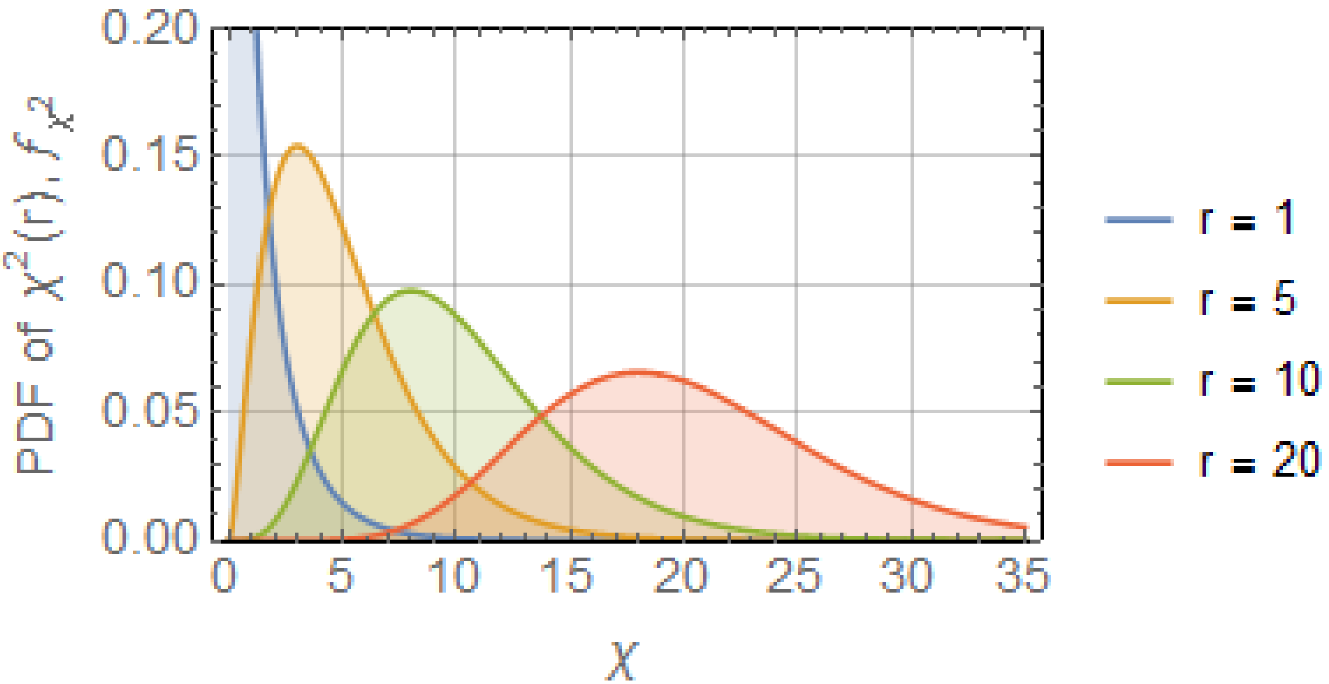
$$f_{\chi^2}(x) = \frac{2^{-r/2} e^{-x/2} x^{\frac{r}{2}-1}}{\Gamma\left(\frac{r}{2}\right)}, \quad x > 0.$$

and

$$E(X) = r$$

$$V(X) = 2r$$

Chi-Sqr probability tables.


0.20
0.15
0.10
0.05
0.00
PDF of $\chi^2(r)$, f_{χ^2}
0
5
10
15
20
25
30
35
x
r = 1
r = 5
r = 10
r = 20

It is important because:

Fact 5.5 If $Y_1, Y_2, \dots, Y_r$ are independent standard normal random variables, $Y_i \sim N(0,1)$, then

$$\sum_{j=1}^{r} Y_j^2 \sim \chi^2(r).$$

Fact 5.6 If $X \sim N(0,1)$, then $Y = X^2 \sim \chi^2(1)$.

Proof Let $Y = X^2$. Then, using the techniques in this chapter

$$f_Y(y) = \frac{1}{2\sqrt{y}}\left(f_X\left(\sqrt{y}\right) + f_X\left(-\sqrt{y}\right)\right) = \frac{1}{\sqrt{2\pi y}} e^{-\frac{1}{2}y}, \quad y > 0.$$

$\blacksquare$

Fact 5.7 If $Y_1 \sim \chi^2(r)$ and $Y_2 \sim \chi^2(s)$, and are independent, then

$$Y_1 + Y_2 \sim \chi^2(r+s).$$

5.3.2 Student's t-distribution

If $U \sim N(0,1)$ and $V \sim \chi^2(r)$ are independent, then

$$T = \frac{U}{\sqrt{V/r}} \sim t(r)$$

has a t-distribution with r degrees of freedom:

$$f_T(t) = \frac{\Gamma\left(\frac{r+1}{2}\right)}{\sqrt{(\pi r)}\Gamma\left(\frac{r}{2}\right)} \left(1 + \frac{t^2}{r}\right)^{-\frac{r+1}{2}}, \ t \in \mathbb{R}.$$

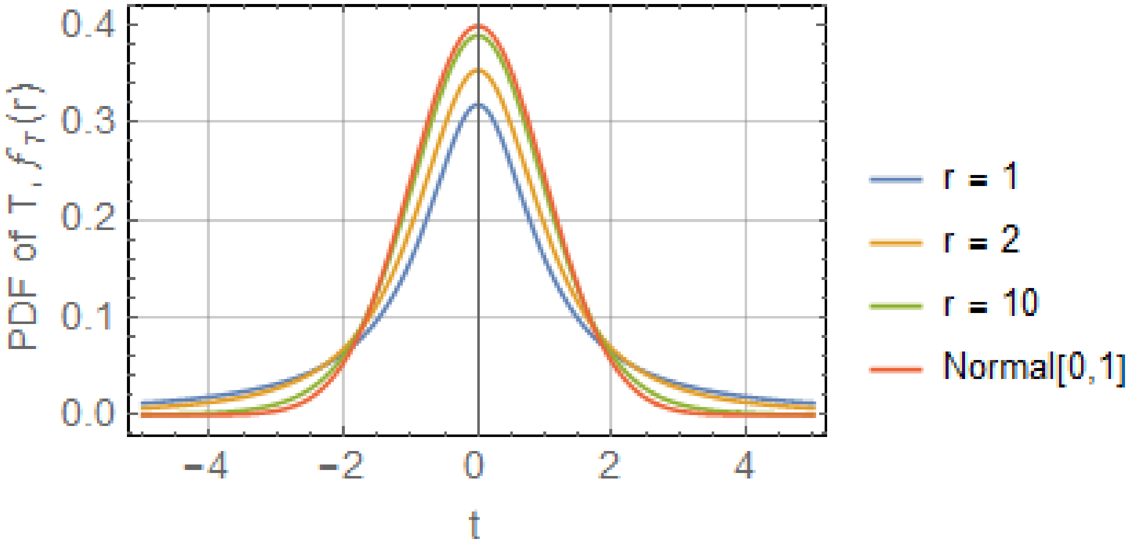
and:

$$\mathrm{E}(T) = 0$$

$$\mathrm{V}(T) = r/(r-2), \quad r > 2$$

Note, as $r \to \infty$ then $t(r) \to \mathrm{N}(0,1)$.

t-distribution tables.


PDF of T , $f_T(r)$
0.4
0.3
0.2
0.1
0.0
-4
-2
0
2
4
t
r = 1
r = 2
r = 10
Normal[0,1]