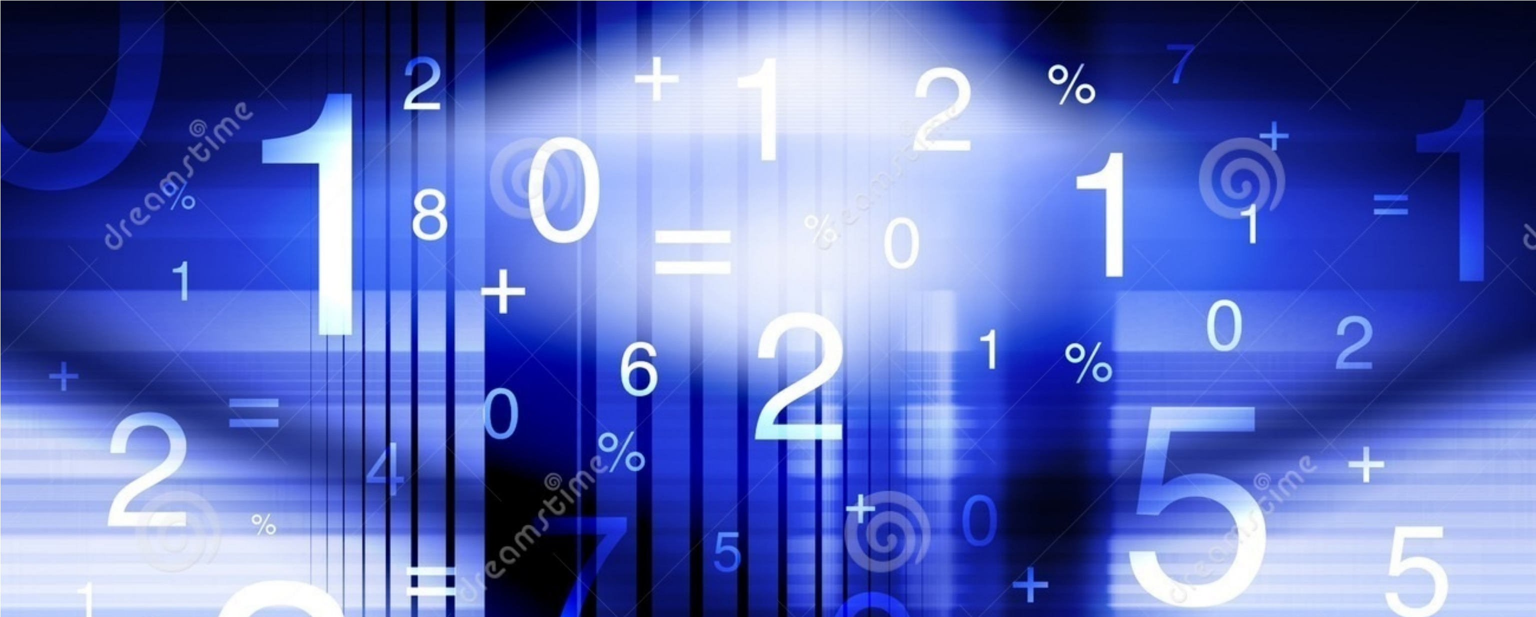
Notes on Probability and Statistics



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Probability

1. Basics of Probability

Probability has its origins in correspondence discussing **the mathematics of games of chance** between Blaise Pascal and Pierre de Fermat in the 17^{th} century, and was formalized and rendered axiomatic as a distinct branch of mathematics by Andrey Kolmogorov in the 20^{th} century.

Two interpretations of probability :

a) Frequentist (this course) If we denote by n_a the number of occurrences of an event A in n trials, then if

$$\boxed{\lim_{n\to+\infty} \frac{n_a}{n} = p} \tag{1.1}$$

we say that the probability of A is p, or $\mathrm{P}(A) = p$.

- Data are a repeatable random sample - there is a frequency
- **Parameters (e.g. p) are fixed and unknown**

b) Bayesian (not covered here)

- Parameters are *random variables*
- **Data are fixed** but can be updated.
- More info...

→ NOVA episode: Prediction by the numbers

1.1 Histograms

For a specific set of experimental data, a histogram shows the relative frequencies of the different observed values of a single variable. They may be constructed as follows.

1. select a range on the x-axis that is sufficient to cover the largest and smallest values among the set of data,
2. divide this range in "convenient" intervals or *bins*.
3. the y-axis can be either (i) the number of observations within each bin among the total number of observations, or (ii) the fraction of the total number.

→ Google image search for histograms in engineering
→ Galton machine video
→ Online histogram maker
→ Google public data

1.2 Terminology and set theory

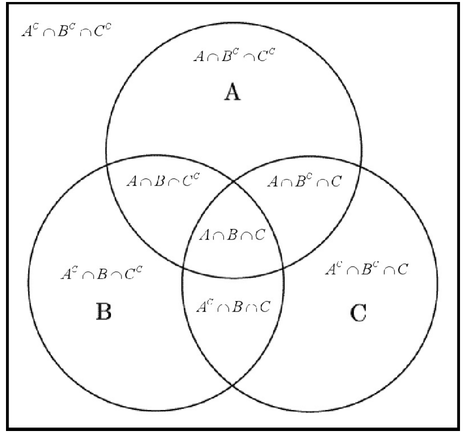
To illustrate the definitions below, we consider the example of a dice roll that generates outcomes in the set $\{1, 2, 3, 4, 5, 6\}$.

Experiment: An action with an uncertain outcome, e.g. a dice roll.

Sample space, S: The set of all possible outcomes of an experiment. In the example. $S = \{1, 2, 3, 4, 5, 6\}$.

Event: Any subset of the sample space, $A \subset S$ would be an event, e.g. $A = \{1, 3, 5\}$. We say that the event has occurred if any of the outcomes in the event has happened.

Elementary event, ω: an event which contains only a single outcome in the sample space, e.g. $\omega = \{2\}$.
aka: basic outcome or simple event.



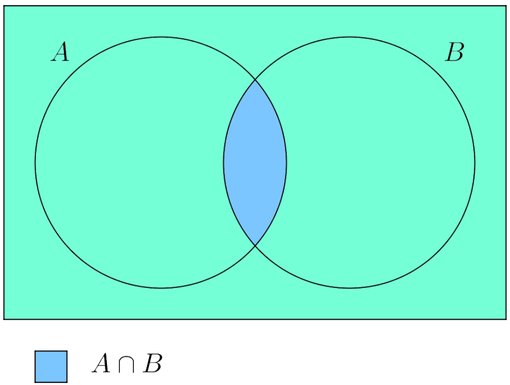
Venn diagrams

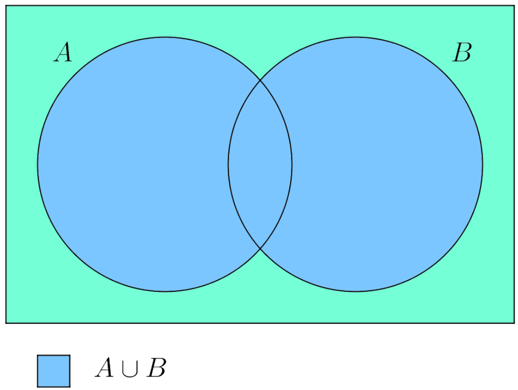
A Venn diagram shows all possible logical relations between a finite collection of different sets. These diagrams depict basic outcomes as points in the plane, and events as regions inside closed curves.

For the definitions below, let A and B be the events $A = \{1,2,3,4\}$ and $B = \{4,5,6\}$.

Intersection of sets, $A \cap B$. The set of all outcomes that are both in A and B is called the intersection of A and B. In the example $A \cap B = \{4\}$

Union of sets, $A \cup B$. The set of all outcomes that are in either of A and B is called the union of A and B. In the example $A \cup B = \{1,2,3,4,5,6\}$





Complement of set, A^c. The set of all outcomes that **are not in** A, but are in S is called the complement of A. In the example $A^c = \{5,6\}$.
Note: $\bar{A}$ and A' are also common notation for complement.

Mutually exclusive (ME) events. Events A and B are mutually exclusive if

$$A \cap B = \emptyset$$

where $\emptyset$ is the empty set.

Collectively exhaustive (CE) events. Events A and B are collectively exhaustive if

$$A \cup B = S$$

Division. Events $A_1, \ldots, A_n$ form a division of the sample space S if

$$\bigcup_{k=1}^{n} A_k = S, \qquad \text{and} \qquad A_i \cap A_j = \emptyset \qquad i \neq j.$$

Notation. For events $A_1, \dots, A_n$:

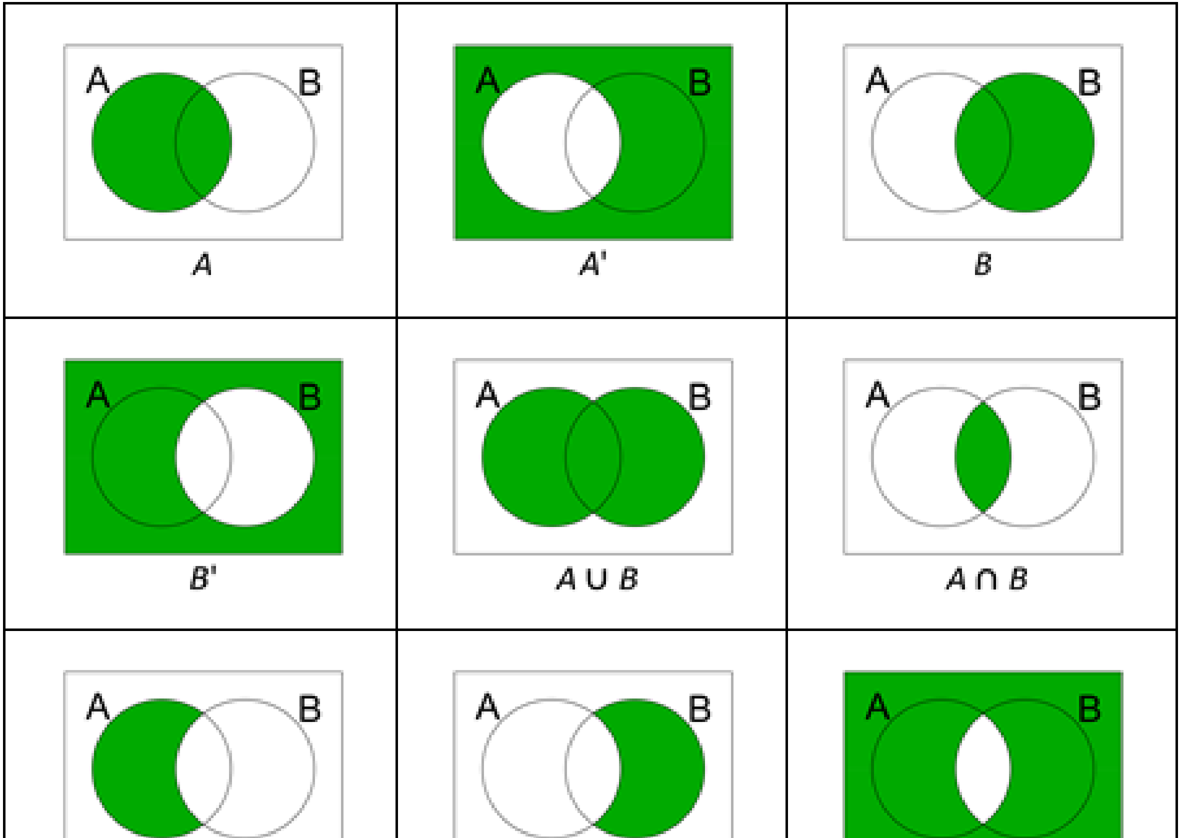
- $\bigcup_{k=1}^{n} A_k = A_1 \cup A_2 \cdots \cup A_n$ and $\bigcap_{k=1}^{n} A_k = A_1 \cap A_2 \cdots \cap A_n$.

  For variables $x_1, \dots, x_n$:

- $\sum_{k=1}^{n} x_k = x_1 + x_2 \cdots + x_n$ and $\prod_{k=1}^{n} x_k = x_1 x_2 \dots x_n$.

→ image source

2 Circle Venn Diagram Shading



Example 1. A coin is tossed twice. Let H stand for heads and T for tails. So:

a) the elementary events are HH, HT, TH and TT

b) the sample space is $S = \{HH, HT, TH, TT\}$

Example 2. Suppose the travel time between two major cities A and B by air is 7 or 8 hr if the flight is nonstop; however, if there is one stop, the travel time would be 10, 11, or 12 hr. A nonstop flight between A and B would cost \$1000, whereas with one stop the cost is only \$650. Then, between cities B and C, all flights are nonstop requiring 2 or 3 hours at a cost of \$250. (There is no flight from A to C)
For a passenger wishing to travel from city A to city C,
(a) What is the possibility space or sample space of his travel times from A to B? From A to C?
(b) What is the sample space of his travel cost from A to B?
(c) If T=travel time from city A to city C, and S=cost of travel from A to C, what is the sample space of T and S?

Solution: (a) Sample space of travel time from A to B = $\{7, 8, 10, 11, 12\}$
Sample space of travel time from A to C = $\{9, 10, 11, 12, 13, 14, 15\}$
(b) Sample space of travel cost from A to B = $\{650, 1000\}$
(c) Sample space of $T = \{9, 10, 11, 12, 13, 14, 15\}$
Sample space of $S = \{900, 1250\}$
Sample space of T and $S = \{\{9, 1250\}, \{10, 1250\}, \{11, 1250\}, \{12, 900\}, \{13, 900\}, \{14, 900\}, \{15, 900\}\}$ □

1.2.1 Basic Laws

Commutative Laws

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

Associative Laws

$$(A \cup B) \cup C = A \cup (B \cup C)$$
$$(A \cap B) \cap C = A \cap (B \cap C)$$

Distributive Laws

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$$
$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$$

Note

$$
\begin{aligned}
A \cup \emptyset &= A \\
A \cap \emptyset &= \emptyset \\
A \cup S &= S \\
A \cap S &= A \\
A \cup A &= A \\
A \cap A &= A \\
(A^c)^c &= A
\end{aligned}
$$

The intersection "is like" multiplication, and the union "is like" addition, but there is no double counting: $A \cup A \neq A$. If there are no parentheses, the intersection takes precedence over the union.

Example 3. — **Simplify:** $(A \cup C)(B \cup C)$

Solution:

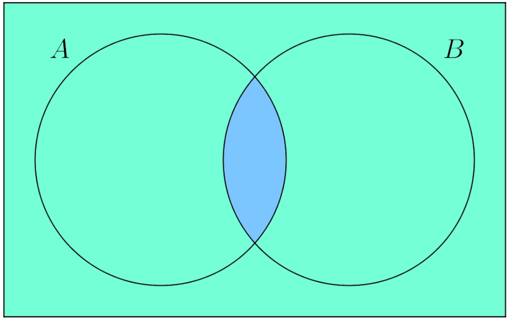
$$\begin{aligned}(A \cup C)(B \cup C) &= AB \cup AC \cup BC \cup CC \\ &= AB \cup AC \cup BC \cup C \\ &= AB \cup AC \cup C \\ &= AB \cup C\end{aligned}$$

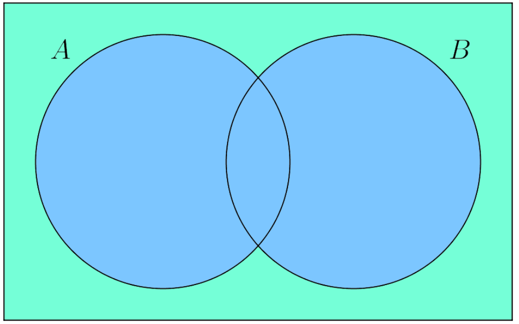
$\square$

De Morgan's Laws

$$(A_1 \cup \cdots \cup A_n)^c = {A_1}^c \cap \cdots \cap {A_n}^c$$
$$(A_1 \cap \cdots \cap A_n)^c = {A_1}^c \cup \cdots \cup {A_n}^c$$

For 2 events:





$A \cap B$

$(A \cap B)^c = A^c \cup B^c$

$A \cup B$

$(A \cup B)^c = A^c \cap B^c$

Not (A and B) is the same as Not A or Not B.
Not (A or B) is the same as Not A and Not B.

Example 4. — A chain Consider a simple chain consisting of two links. The chain will fail to carry a given load if either link breaks. Let:
A = the breakage of link 1
B = the breakage of link 2
Then,

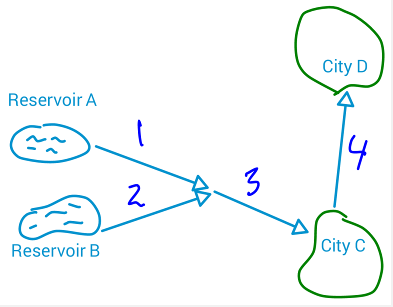
the chain fails = $A \cup B$

No failure of the chain, therefore, is the complement $(A \cup B)^c$. However, no failure of the chain also means that both links survive (no breakage); that is,

the chain does not fail = $A^c \cap B^c$

which is a demonstration of the validity of De Morgan's rule.

Example 5. — Water supply system The water supply for two cities C and D comes from the two sources A and B as shown in the figure. Water is transported by pipelines consisting of branches 1, 2, 3, and 4. Assume that either one of the two sources, by itself, is sufficient to supply the water for both cities.



Denote: E_i = failure of branch $i = 1, 2, 3, 4$. Failure of a pipe branch means there is serious leakage or rupture of the branch. Using these events, express the events

a) water shortage in city C
b) no water shortage in city C (simplify using De Morgan)
c) water shortage in city D
d) no water shortage in city D

Solution: We have:

a) water shortage in city C: $E_1E_2 \cup E_3$

b) no water shortage in city C: $(E_1E_2 \cup E_3)^c = (E_1E_2)^c {E_3}^c = ({E_1}^c \cup {E_2}^c){E_3}^c$

c) water shortage in city D: $E_1E_2 \cup E_3 \cup E_4$

d) no water shortage in city D: $(E_1E_2 \cup E_3 \cup E_4)^c = ({E_1}^c \cup {E_2}^c){E_3}^c{E_4}^c$

$\square$

1.3 Combinatorics: counting strategies when outcomes are equally likely

Motivation for combinatorics:

Fact 1.7 — Probability when outcomes are equally likely. If all outcomes of an experiment are equally likely, the probability of an event A happening is:

$$\mathrm{P}(A) = \frac{\text{number of outcomes favorable to } A}{\text{number of outcomes}} = \frac{|A|}{|S|}$$

where $|A|$ is the size of set A.

$\rightarrow$ Combinatorics helps calculate $|A|$ and $|S|$.

Example 6. At **Coco's restaurant** you have

a) two choices for appetizers: soup or juice;
b) three for the main course: a meat, fish, or vegetable dish; and
c) two for dessert: ice cream or cake.

How many possible choices do you have for your complete meal?

Solution: We illustrate the possible meals by a tree diagram, where we see that the total number of choices is the product of the number of choices at each stage. In this examples we have $2 \cdot 3 \cdot 2 = 12$ possible menus. $\square$

Our menu example illustrates the following general counting technique.

Basic Rule in combinatorics In an experiment with k steps, if

- the 1st step has n_1 possible outcomes,
- the 2nd step has n_2 possible outcomes,
- ...
- the kth step has n_k possible outcomes, then there are:

$$\boxed{n_1 \times n_2 \times \cdots \times n_k} \tag{1.2}$$

possible outcomes for the whole experiment.

Example 7. A finite set Ω has n elements. Show that if we count the empty set and Ω as subsets, there are 2^n subsets of Ω.

Solution: The experiment of generating a subset of Ω can be broken down in n steps, one for each element in Ω, and for each element we have 2 choices: we either pick or do not pick the element. $\square$

Example 8. — Permutations. The English alphabet has 26 letters. How many 5-letter "**words**" are there if:

a) repeated letters are allowed (experiment with replacement)

b) repeated letters are not allowed (without replacement)

Solution:

a) repeated letters are allowed: $26 \times 26 \times 26 \times 26 \times 26 = 26^5$

b) repeated letters are not allowed: $26 \times 25 \times 24 \times 23 \times 22$

$\square$

Example 9. In example 8, how many 5-letter "**words**" contain NO letter "S", if:

a) repeated letters are allowed (with replacement)

b) repeated letters are not allowed (without replacement)

Solution:

a) repeated letters are allowed: 25^5

b) repeated letters are not allowed: $25 \times 24 \times 23 \times 22 \times 21$

$\square$

Example 10. In example 8, how many 5-letter "**words**" contain at least 1 letter "S", if:

a) repeated letters are allowed (with replacement)

b) repeated letters are not allowed (without replacement)

Solution:

a) repeated letters are allowed: $26^5 - 25^5$

b) repeated letters are not allowed: $26 \times 25 \times 24 \times 23 \times 22 - 25 \times 24 \times 23 \times 22 \times 21$

$\square$

Example 11. In example 8, how many 5-letter "**words**" contain exactly one letter "S", if:

a) repeated letters are allowed (with replacement)

b) repeated letters are not allowed (without replacement)

Solution:

a) repeated letters are allowed: 5×25^4

(first step: place letter S in any of the five spots, second step: fill any of the four spots available with any of the 25 remaining letters,...)

b) repeated letters are not allowed: $5 \times 25 \times 24 \times 23 \times 22$

$\square$

Example 12. In example 8, how many 5-letter "**words**" contain exactly 1 letter "S" and 1 letter "O", if:

a) repeated letters are allowed (with replacement)

b) repeated letters are not allowed (without replacement)

Solution:

a) repeated letters are allowed: $5 \times 4 \times 24^3$
(first step: place letter S in any of the five spots, second step: place letter O in any of the four remaining spots, third step: fill any of the 3 remaining spots with any of the 24 remaining letters,...)

b) repeated letters are not allowed: $5 \times 4 \times 24 \times 23 \times 22$

$\square$

Ordered sequence (word) is a list of elements where the order matters, e.g. $\{1,2,3,2\} \neq \{1,2,2,3\}$.

Permutations: The number of distinct **ordered sequences** with k elements that can be chosen from a set with n elements.

Fact 1.9 The number of **permutations with replacement** of k objects out of n is

$$n^k$$

and gives the number of *ordered sequences* possible when selecting k objects out of n *with replacement.*

Fact 1.10 The number of **permutations without replacement** of k objects out of n,

$$^{n}\mathrm{P}_k = n(n-1)(n-2)\ldots(n-k+1) = \frac{n!}{(n-k)!}$$

gives the number of *ordered sequences* possible when selecting k objects out of n *without replacement.*

Example 13. — Combinations. How many **groups** of 3 students can be made in a class of 4 students?

Solution: There are ${}^4\mathrm{P}_3$ **ordered** sequences of 3 students:

1	1	**1**	1	1	1	**2**	2	**2**	2	2	2	**3**	3	**3**	3	3	3	4	4	4	4	4	4
2	2	**3**	3	4	4	**1**	1	**3**	3	4	4	**1**	1	**2**	2	4	4	1	1	2	2	3	3
3	4	**2**	4	2	3	**3**	4	**1**	4	1	3	**2**	4	**1**	4	1	2	2	3	1	3	1	2

Since 3 students can be shuffled in $3! = 6$ different ways, the answer is ${}^4\mathrm{P}_3/3!$. $\square$

An unordered sequence (group) is a list of elements where the order DOES NOT matters, e.g. the group $\{1,2,3,4\}$ is equivalent to $\{1,4,2,3\}$.

Combinations: The number of distinct **unordered sequences** with k elements that can be chosen, without replacement, from a set with n elements is denoted by $\binom{n}{k}$, and is pronounced "n choose k." The number $\binom{n}{j}$ is called a binomial coefficient.

Fact 1.11 The number of **combinations without replacement** of k objects out of n,

$$\binom{n}{k} = {}^{n}\mathrm{P}_{k}/k! = \frac{n!}{(n-k)!k!}$$

gives the number of *unordered sequences* possible when selecting k objects out of n *without replacement.*
Note: $\binom{n}{k} = \binom{n}{n-k}$

The sampling table gives the number of possible samples (sequences) of size k out of a population of size n, depending on how the sample is collected.

	Permutations (order matters)	**Combinations (not matter)**
With Replacement	n^k	$\binom{n+k-1}{k}$
Without Replacement	${}^{n}\mathrm{P}_k = \frac{n!}{(n-k)!}$	$\binom{n}{k}$

$\rightarrow$ More info on combinations with replacement...

Example 14. — Georgia lottery Powerball Pick 6 different numbers:

- 5 between 1-69 and
- 1 PowerBall number between 1-26.

what are the chances of winning?

Solution: $|S| = \binom{69}{5}\binom{26}{1} = 292,201,338 \to$ P(winning)=1/292,201,338. $\square$

Example 15. * There are n students in a classroom. Assuming 365 days in a year, what is the probability that everyone has distinct birthdays , ignoring the year?

Solution: There are 365 options for each person's birthday. The sample space is all possible birthday sequences of length n. Therefore,

$$|S| = 365^n$$

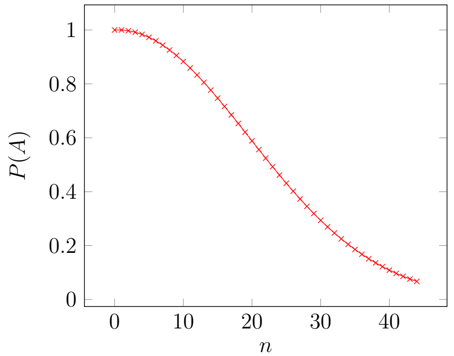
Let A be the event that everyone has distinct birthdays.
If $n > 365$, at least two persons must share a birthday.

If $n \leq 365$,

$$|A| = 365 \times 364 \ldots (365 - n + 1) = {}^{365}\mathrm{P}_n$$

Therefore,

$$\begin{aligned} \mathrm{P}(A) &= \frac{{}^{365}\mathrm{P}_n}{365^n} \\ &= \frac{\frac{365!}{(365-n)!}}{365^n} \end{aligned}$$



NOTE: the probability of at least two persons sharing a birthday is A^c, and

$$|A^c| = |S| - |A|, \qquad \text{and therefore:}$$

$$\boxed{\mathrm{P}(A^c) = \frac{|S| - |A|}{|S|} = 1 - \mathrm{P}(A)} \tag{1.3}$$

$\square$

Important note on identical objects: Consider n objects of which k are identical of type A and the remaining $(n-k)$ of type B. The number of ways one can arrange these n objects is $\binom{n}{k} = \frac{n!}{(n-k)!k!}$.

Why? label each of the n objects with a unique number so that all objects are different. These n objects can be arranged in $n!$ different ways. If we remove the labels, we would see that each particular arrangement is repeated $k!(n-k)!$ times (recall that k distinct objects can be ordered in $k!$ ways).
For example, the number of arrangements of 3 A's and 2 B's **after labeling** is 5!=120:

$\{A_1,A_2,A_3,B_1,B_2\}$	$\{A_1,A_2,A_3,B_2,B_1\}$	$\{A_1,A_2,B_1,A_3,B_2\}$	$\{A_1,A_2,B_1,B_2,A_3\}$	$\{A_1,A_2,B_2,A_3,B_1\}$
$\{A_1,A_2,B_2,B_1,A_3\}$	$\{A_1,A_3,A_2,B_1,B_2\}$	$\{A_1,A_3,A_2,B_2,B_1\}$	$\{A_1,A_3,B_1,A_2,B_2\}$	$\{A_1,A_3,B_1,B_2,A_2\}$
$\{A_1,A_3,B_2,A_2,B_1\}$	$\{A_1,A_3,B_2,B_1,A_2\}$	$\{A_1,B_1,A_2,A_3,B_2\}$	$\{A_1,B_1,A_2,B_2,A_3\}$	$\{A_1,B_1,A_3,A_2,B_2\}$
$\{A_1,B_1,A_3,B_2,A_2\}$	$\{A_1,B_1,B_2,A_2,A_3\}$	$\{A_1,B_1,B_2,A_3,A_2\}$	$\{A_1,B_2,A_2,A_3,B_1\}$	$\{A_1,B_2,A_2,B_1,A_3\}$
$\{A_1,B_2,A_3,A_2,B_1\}$	$\{A_1,B_2,A_3,B_1,A_2\}$	$\{A_1,B_2,B_1,A_2,A_3\}$	$\{A_1,B_2,B_1,A_3,A_2\}$	$\{A_2,A_1,A_3,B_1,B_2\}$
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$\{B_1,A_2,A_3,A_1,B_2\}$	$\{B_1,A_2,A_3,B_2,A_1\}$	$\{B_1,A_2,B_2,A_1,A_3\}$	$\{B_1,A_2,B_2,A_3,A_1\}$	$\{B_1,A_3,A_1,A_2,B_2\}$
$\{B_1,A_3,A_1,B_2,A_2\}$	$\{B_1,A_3,A_2,A_1,B_2\}$	$\{B_1,A_3,A_2,B_2,A_1\}$	$\{B_1,A_3,B_2,A_1,A_2\}$	$\{B_1,A_3,B_2,A_2,A_1\}$
$\{B_1,B_2,A_1,A_2,A_3\}$	$\{B_1,B_2,A_1,A_3,A_2\}$	$\{B_1,B_2,A_2,A_1,A_3\}$	$\{B_1,B_2,A_2,A_3,A_1\}$	$\{B_1,B_2,A_3,A_1,A_2\}$
$\{B_1,B_2,A_3,A_2,A_1\}$	$\{B_2,A_1,A_2,A_3,B_1\}$	$\{B_2,A_1,A_2,B_1,A_3\}$	$\{B_2,A_1,A_3,A_2,B_1\}$	$\{B_2,A_1,A_3,B_1,A_2\}$
$\{B_2,A_1,B_1,A_2,A_3\}$	$\{B_2,A_1,B_1,A_3,A_2\}$	$\{B_2,A_2,A_1,A_3,B_1\}$	$\{B_2,A_2,A_1,B_1,A_3\}$	$\{B_2,A_2,A_3,A_1,B_1\}$
$\{B_2,A_2,A_3,B_1,A_1\}$	$\{B_2,A_2,B_1,A_1,A_3\}$	$\{B_2,A_2,B_1,A_3,A_1\}$	$\{B_2,A_3,A_1,A_2,B_1\}$	$\{B_2,A_3,A_1,B_1,A_2\}$
$\{B_2,A_3,A_2,A_1,B_1\}$	$\{B_2,A_3,A_2,B_1,A_1\}$	$\{B_2,A_3,B_1,A_1,A_2\}$	$\{B_2,A_3,B_1,A_2,A_1\}$	$\{B_2,B_1,A_1,A_2,A_3\}$
$\{B_2,B_1,A_1,A_3,A_2\}$	$\{B_2,B_1,A_2,A_1,A_3\}$	$\{B_2,B_1,A_2,A_3,A_1\}$	$\{B_2,B_1,A_3,A_1,A_2\}$	$\{B_2,B_1,A_3,A_2,A_1\}$

If we remove the labels, we see that each pattern is repeated 3!2!=12 times:

$\{A,A,A,B,B\}*$	$\{A,A,A,B,B\}*$	$\{A,A,B,A,B\}$	$\{A,A,B,B,A\}$	$\{A,A,B,A,B\}$
$\{A,A,B,B,A\}$	$\{A,A,A,B,B\}*$	$\{A,A,A,B,B\}*$	$\{A,A,B,A,B\}$	$\{A,A,B,B,A\}$
$\{A,A,B,A,B\}$	$\{A,A,B,B,A\}$	$\{A,B,A,A,B\}$	$\{A,B,A,B,A\}$	$\{A,B,A,A,B\}$
$\{A,B,A,B,A\}$	$\{A,B,B,A,A\}$	$\{A,B,B,A,A\}$	$\{A,B,A,A,B\}$	$\{A,B,A,B,A\}$
$\{A,B,A,A,B\}$	$\{A,B,A,B,A\}$	$\{A,B,B,A,A\}$	$\{A,B,B,A,A\}$	$\{A,A,A,B,B\}*$
$\{A,A,A,B,B\}*$	$\{A,A,B,A,B\}$	$\{A,A,B,B,A\}$	$\{A,A,B,A,B\}$	$\{A,A,B,B,A\}$
$\{A,A,A,B,B\}*$	$\{A,A,A,B,B\}*$	$\{A,A,B,A,B\}$	$\{A,A,B,B,A\}$	$\{A,A,B,A,B\}$
$\{A,A,B,B,A\}$	$\{A,B,A,A,B\}$	$\{A,B,A,B,A\}$	$\{A,B,A,A,B\}$	$\{A,B,A,B,A\}$
$\{A,B,B,A,A\}$	$\{A,B,B,A,A\}$	$\{A,B,A,A,B\}$	$\{A,B,A,B,A\}$	$\{A,B,A,A,B\}$
$\{A,B,A,B,A\}$	$\{A,B,B,A,A\}$	$\{A,B,B,A,A\}$	$\{A,A,A,B,B\}*$	$\{A,A,A,B,B\}*$
$\{A,A,B,A,B\}$	$\{A,A,B,B,A\}$	$\{A,A,B,A,B\}$	$\{A,A,B,B,A\}$	$\{A,A,A,B,B\}*$
$\{A,A,A,B,B\}*$	$\{A,A,B,A,B\}$	$\{A,A,B,B,A\}$	$\{A,A,B,A,B\}$	$\{A,A,B,B,A\}$
$\{A,B,A,A,B\}$	$\{A,B,A,B,A\}$	$\{A,B,A,A,B\}$	$\{A,B,A,B,A\}$	$\{A,B,B,A,A\}$
$\{A,B,B,A,A\}$	$\{A,B,A,A,B\}$	$\{A,B,A,B,A\}$	$\{A,B,A,A,B\}$	$\{A,B,A,B,A\}$
$\{A,B,B,A,A\}$	$\{A,B,B,A,A\}$	$\{B,A,A,A,B\}$	$\{B,A,A,B,A\}$	$\{B,A,A,A,B\}$
$\{B,A,A,B,A\}$	$\{B,A,B,A,A\}$	$\{B,A,B,A,A\}$	$\{B,A,A,A,B\}$	$\{B,A,A,B,A\}$
$\{B,A,A,A,B\}$	$\{B,A,A,B,A\}$	$\{B,A,B,A,A\}$	$\{B,A,B,A,A\}$	$\{B,A,A,A,B\}$
$\{B,A,A,B,A\}$	$\{B,A,A,A,B\}$	$\{B,A,A,B,A\}$	$\{B,A,B,A,A\}$	$\{B,A,B,A,A\}$
$\{B,B,A,A,A\}$	$\{B,B,A,A,A\}$	$\{B,B,A,A,A\}$	$\{B,B,A,A,A\}$	$\{B,B,A,A,A\}$
$\{B,B,A,A,A\}$	$\{B,A,A,A,B\}$	$\{B,A,A,B,A\}$	$\{B,A,A,A,B\}$	$\{B,A,A,B,A\}$
$\{B,A,B,A,A\}$	$\{B,A,B,A,A\}$	$\{B,A,A,A,B\}$	$\{B,A,A,B,A\}$	$\{B,A,A,A,B\}$
$\{B,A,A,B,A\}$	$\{B,A,B,A,A\}$	$\{B,A,B,A,A\}$	$\{B,A,A,A,B\}$	$\{B,A,A,B,A\}$
$\{B,A,A,A,B\}$	$\{B,A,A,B,A\}$	$\{B,A,B,A,A\}$	$\{B,A,B,A,A\}$	$\{B,B,A,A,A\}$
$\{B,B,A,A,A\}$	$\{B,B,A,A,A\}$	$\{B,B,A,A,A\}$	$\{B,B,A,A,A\}$	$\{B,B,A,A,A\}$

So there are only $\binom{n}{k} = \binom{5}{3}$=5!/(3!2!)=10 different patterns:

$\{A,A,A,B,B\}$	$\{A,A,B,A,B\}$	$\{A,A,B,B,A\}$	$\{A,B,A,A,B\}$	$\{A,B,A,B,A\}$
$\{A,B,B,A,A\}$	$\{B,A,A,A,B\}$	$\{B,A,A,B,A\}$	$\{B,A,B,A,A\}$	$\{B,B,A,A,A\}$

In general:

Several groups of identical objects: Consider n_i identical objects of type $i = 1, 2, \ldots m$. The number of ways one can arrange these $n = \sum_{i=1}^{m} n_i$ objects is

$$\frac{n!}{n_1!n_2!\ldots n_m!}$$

Example 16. A *Bernoulli* trial is an experiment that can result in either a success or a failure. In how many ways can the results of n Bernoulli trials be arranged such that there are k successes?

Solution: $\binom{n}{k}$ $\square$

Example 17. The Georgia Tech football team played 8 games in a season, winning 3, losing 3, and ending 2 in a tie. In how many ways could this have happened?

Solution: $\frac{8!}{3!\,3!\,2!}$ $\square$

Example 18. — Books on a shelf In how many ways can we arrange 3 books on a shelf with capacity for 4 books?

Solution: $|A| = \binom{4}{3} = 4$ $\square$

Example 19. A nickel is tossed 4 times. What is the probability of obtaining 3 heads?

Solution: $|\mathrm{S}|=2^4=16$
The number of outcomes that yield 3H is identical to the number of ways we can arrange 3 books on a shelf with capacity for 4 books, so:
$|A| = \binom{4}{3} = 4$ and $\mathrm{P}(A) = 4/16$. $\square$

Example 20. A nickel is tossed 4 times. What is the probability of obtaining 3 heads if the coin is biased with a probability of heads of 0.53?

Solution: Since the coin is biased outcomes are not equally likely and therefore combinatorics techniques cannot be applied. Later we will see that this is given by the binomial distribution. $\square$

Example 21. 8 books are to be arranged on 2 shelves, of capacities 3 and 5 respectively. Out of the 8 books, 2 books are special. Find the probability that the two special books end up on the same shelf.

Solution: [1] If the special books are to be placed on the longer shelf, the possible combinations are $\binom{5}{2}$.
If the special books are to be placed on the shorter shelf, the possible combinations are $\binom{3}{2}$.
To total possible arrangements are $\binom{8}{2}$.
Therefore, the required probability is $\frac{\binom{5}{2}+\binom{3}{2}}{\binom{8}{2}} = 13/28$. $\square$

Solution: [2] Let the special books be placed first.
If the first special book is placed on the longer shelf, then it has 5 available positions, and the second special book has 4 available positions.
If the first special book is placed on the shorter shelf, then it has 3 available positions, and the second special book has 2 available positions.
In either case, the number of ways of arranging the remaining 6 books in the remaining positions is 6!.
Therefore, the total number of arrangements satisfying the conditions is $(5 \cdot 4 \cdot 6! + 3 \cdot 2 \cdot 6!)$. The total number of arrangements is 8!. Therefore, the required probability is $\frac{5 \cdot 4 \cdot 6! + 3 \cdot 2 \cdot 6!}{8!} = 13/28$. $\square$

Example 22. — * The state of GA has license plates showing three numbers and four letters. How many different license plates are possible

a) if the numbers must come after the letters?
b) if there is no restriction on where the letters and numbers appear?
c) as in part b) but replacing all numbers by an "A" and all letters by "B"?
d) BONUS: as in part b) but replacing all numbers < 5 by an "A", all numbers ≥ 5 by "B" and all letters by "C"?

Solution:

a) $10^3 \times 26^4$
b) There are $\binom{7}{3}$ possible arrangements of letters and numbers, each having the same number of outcomes as in part a), so the answer is $\binom{7}{3} \times 10^3 \times 26^4$.
c) $\binom{7}{3}$
d) $\sum_{n_A=0}^{3} \frac{7!}{n_A!(3-n_A)!4!}$

$\square$

Example 23. Letters and mailboxes.]

a) How many ways can six indistinguishable letters be put in three mail boxes?

b) Using part a) above, show that r indistinguishable objects can be put in n boxes in

$$\binom{n+r-1}{n-1} = \binom{n+r-1}{r}$$

different ways.

Solution:

a) One representation of this is given by the sequence |LL|L|LLL| where the |'s represent the partitions for the boxes and the L's the letters. Any possible way can be so described. Note that we need two bars at the ends and the remaining two bars and the six L's can be put in any order. In this way, the problem boils down to shuffling six identical objects and two identical objects, therefore the answer is $\binom{8}{2}$ or $\binom{8}{6}$. Both give the same answer

b) same logic as above using r letters n mailboxes.

$\square$

1.3.1 More problems in combinatorics (optional)

Example 24. Three balls are to be randomly selected, without replacement, from an urn containing 20 balls numbered 1 to 20. If Alice bets that at least one of the balls drawn has a number as large as or larger than 17, what is the probability that Alice wins the bet?

Solution: Let X be the largest number selected.
Therefore, X is a random variable which has a value from $\{3,\ldots,20\}$.
Let the value of the highest valued ball be i. Therefore, the number of ways to select the remaining two balls is $\binom{i-1}{2}$.
Therefore, the probability of the value of the highest valued ball being i is

$$P(X=i)=\frac{\binom{i-1}{2}}{\binom{20}{3}}$$

Therefore,

$$\begin{aligned}P(X\geq 17)&=P(X=17)+P(X=18)+P(X=19)+P(X=20)\\&=\frac{\binom{16}{2}+\binom{17}{2}+\binom{18}{2}+\binom{19}{2}}{\binom{20}{3}}\end{aligned}$$

$\square$

Example 25. A president, a treasurer, and a secretary, all different, are to be chosen from a club consisting of 10 people. How many different choices of office bearers are possible if

1. There are no restrictions.
2. Alice and Bob cannot serve together.
3. Charlie and David can serve together or not at all.
4. Eve must be an officer.
5. Frank can serve only if he is the president.

Solution:

1. The possible choices are ${}^{10}\mathrm{P}_3$.
2. If neither Alice nor Bob are office bearers, there are ${}^{8}\mathrm{P}_3$ possible choices.
   If one of Alice and Bob is an office bearer, there are three possible posts for the selected person. The number of choices for the rest of the posts are ${}^{8}\mathrm{P}_2$. Same for Bob. Therefore, the total number of choices are ${}^{8}\mathrm{P}_3 + 2\cdot 3\cdot {}^{8}\mathrm{P}_2$.
3. If both Charlie and David are chosen, the number of choices is $3\cdot 2\cdot {}^{8}\mathrm{P}_1$. If neither Charlie nor David are chosen, the number of choices is ${}^{8}\mathrm{P}_3$. Therefore, the total number of choices are $3\cdot 2\cdot \binom{8}{1} + {}^{8}\mathrm{P}_3$.
4. There are three possible posts for Eve. Therefore, the total number of choices are $3\cdot {}^{9}\mathrm{P}_2$.
5. If Frank is the president, the number of choices is ${}^{9}\mathrm{P}_2$. If Frank is not the president, the number of choices is ${}^{9}\mathrm{P}_3$. Therefore, the total number of choices is ${}^{9}\mathrm{P}_2 + {}^{9}\mathrm{P}_3$.

$\square$

Example 26. a different balls are divided randomly into n different cells. Find the probability that all cells are non-empty when

a) $a = n$

b) $a = n + 1$

Solution: We have:

a)

$$\begin{aligned}|S| &= n^a \\ &= n^n\end{aligned}$$

If all cells are to be non-empty, the number of combinations is $|A| = n!$. Therefore, the probability is $\frac{n!}{n^n}$.

b)

$$\begin{aligned}|S| &= n^a \\ &= n^{n+1}\end{aligned}$$

The number of combinations to select 2 balls is $\binom{n+1}{2}$. Let these two balls be glued together and be treated as one.

The number of arrangements of these n balls are $n!$.

Therefore, the total number of combinations are $\binom{n+1}{2} n!$.

Therefore, the probability is $\frac{\binom{n+1}{2}}{n!}$.

$\square$

Example 27. — Poker: why a four of a kind beats a full house? A poker hand is a random subset of 5 elements from a deck of 52 cards.

a) How many hands have four of a kind?

b) How many hands have a full house?

Solution:

a) How many hands have four of a kind? There are 13 ways that we can specify the value for the four cards. For each of these, there are 48 possibilities for the fifth card. Thus, the number of four-of-a-kind hands is $13 \cdot 48 = 624$. Since the total number of possible hands is $\binom{52}{5} = 2598960$, the probability of a hand with four of a kind is $624/2598960 = .00024$.

b) Full house: There are 13 choices for the value which occurs three times; for each of these there are $\binom{4}{3} = 4$ choices for the particular three cards of this value that are in the hand. Having picked these three cards, there are 12 possibilities for the value which occurs twice; for each of these there are $\binom{4}{2} = 6$ possibilities for the particular pair of this value. Thus, the number of full houses is $13 \cdot 4 \cdot 12 \cdot 6 = 3744$, and the probability of obtaining a hand with a full house is $3744/2598960 = .0014$.

Thus, while both types of hands are unlikely, you are six times more likely to obtain a full house than four of a kind. $\square$

Example 28. 5 cards are taken out randomly from a 52 card deck. Consider the following events.

a) A: All cards are higher than 10.

b) B: All cards are hearts.

c) C: All cards have different numbers.

d) D: All cards are consecutive numbers.

Assuming ace to have value 1, find the probabilities of A, B, C, and D.

Solution:

$$|S| = \binom{52}{5}$$

a) There are 12 cards with numbers higher than 10. Therefore,

$$|A| = \binom{12}{5}$$

Therefore,

$$\mathrm{P}(A) = \frac{|A|}{|S|} = \frac{\binom{12}{5}}{\binom{52}{5}}$$

b) There are 13 heart cards. Therefore,

$$|B| = \binom{13}{5}$$

Therefore,

$$\mathrm{P}(B) = \frac{|B|}{|S|} = \frac{\binom{13}{5}}{\binom{52}{5}}$$

c) There are $52 \times 48 \times 44 \times 40 \times 36$ ways to have all cards with different numbers, but since order dos not matter:

$$|C| = 52 \times 48 \times 44 \times 40 \times 36/5! = 1,317,888$$

Alternatively, the number of ways of selecting 5 different numbers out of 13 is $\binom{13}{5}$. For each of the selected number, there are 4 cards, of which exactly one has to be selected. Therefore,

$$|C| = \binom{13}{5} 4^5 = 1,317,888$$

Therefore,

$$\mathrm{P}(C) = \frac{1,317,888}{\binom{52}{5}}$$

d) There are 9 sequences of consecutive numbers. Each of the numbers have 4 corresponding cards each. Therefore,

$$|D| = 9 \cdot 4^5$$

Therefore,

$$\mathrm{P}(D) = \frac{9 \cdot 4^5}{\binom{52}{3}}$$

$\square$

Example 29. A dice is tossed 3 times. Consider the following events.

a) A: The sum of all three numbers is even.

Find the probability of A.

Solution: Every time the dice is rolled, there as 6 possible outcomes. Therefore,

$$|S| = 6^3$$

a) For the sum of three numbers to be even, exactly 0 or 2 of them must be odd.
There is $\binom{3}{3} = 1$ combination for all three numbers to be even. Each of these even numbers has 3 options. Therefore, the total number of combinations following the restriction in 1×3^3.
There are $\binom{3}{2} = 3$ combinations for exactly 2 numbers to be odd. Each of the odd numbers has 3 option, and the even number has 3 options. Therefore, the total number of combinations following the restriction in 3×3^3.
Therefore, $|A| = 3^3 + 3^4$ and $\mathrm{P}(A) = \frac{3^3+3^4}{6^3}$. $\square$

1.4 Axioms of Probability

Probability: The probability of an event E, denoted by $\mathrm{P}(E)$, is a function that satisfies the three basic axioms:in urban

Axiom 1:

$$0 \leq \mathrm{P}(E) \leq 1$$

Axiom 2:

$$\mathrm{P}(S) = 1$$

Axiom 3: For any sequence of mutually exclusive events $A_1,\ A_2 \ldots,$

$$\mathrm{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathrm{P}(A_i)$$

Fact 1.12

$$P(\emptyset) = 0$$

Fact 1.13 For a finite collection of mutually exclusive event $A_1, \ldots, A_n$,

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$$

Probability of the complement

$$P(A^c) = 1 - P(A)$$

Proof.

$$A \cap A^c = \emptyset$$

Therefore, A and A^c are mutually exclusive. Therefore,

$$\begin{aligned}
\mathrm{P}(A)+\mathrm{P}\left(A^c\right) &= \mathrm{P}\left(A\cup A^c\right)\\
&= \mathrm{P}(S)\\
&= 1\\
\therefore \mathrm{P}\left(A^c\right) &= 1-\mathrm{P}(A)
\end{aligned}$$

■

Example 30. — Three traffic signals A lady crosses three traffic signals, with red and green lights only, on the way to her dog's hairdresser. The probabilities of encountering 0, 1, and 2 red lights are 0.4, 0.1, 0.2 respectively.
Find the probabilities of

a) Encountering at least one red light.
b) Encountering at least one green light.
c) Encountering an odd number of red lights.

Solution:

a) Encountering at least one red light.

$$\begin{aligned}\mathrm{P}(1\text{ red})+\mathrm{P}(2\text{ red})+\mathrm{P}(3\text{ red}) &= 1-\mathrm{P}(0\text{ red})\\ &= 1-0.4\\ &= 0.6\end{aligned}$$

b) Encountering at least one green light.

$$\begin{aligned}\mathrm{P}(1\text{ green})+\mathrm{P}(2\text{ green})+\mathrm{P}(3\text{ green}) &= \mathrm{P}(0\text{ red})+\mathrm{P}(1\text{ red})+\mathrm{P}(2\text{ red})\\ &= 0.4+0.1+0.2\\ &= 0.7\end{aligned}$$

c) Encountering an odd number of red lights.

$$\begin{aligned}\mathrm{P}(1\text{ red})+\mathrm{P}(3\text{ red}) &= 1-\mathrm{P}(0\text{ red})-\mathrm{P}(2\text{ red})\\ &= 1-0.4-0.2\\ &= 0.4\end{aligned}$$

$\square$

1.5 Addition rule

Suppose that you have two finite sets A and B. We can find the size of their union using

$$|A \cup B| = |A| + |B| - |A \cap B|$$

because when you work out $|A| + |B|$ the elements of $A \cap B$ are being 'counted twice'. You compensate for this by subtracting $|A \cap B|$.

Fact 1.15 — Addition rule (aka inclusion-exclusion formula).

$$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$$

Example 31. Let A be the event "even number" and B the event "number>3" when a fair dice is thrown. $\mathrm{P}(A \cup B)$?

Solution: $\mathrm{P}(A) = \mathrm{P}(B) = \frac{3}{6}$ and $\mathrm{P}(A \cap B) = \frac{2}{6}$. Hence $\mathrm{P}(A \cup B) = \frac{3}{6} + \frac{3}{6} - \frac{2}{6} = \frac{4}{6}$, i.e. $\mathrm{P}(\{2,4,5,6\}) = \frac{2}{3}$. $\square$

Example 32. Ben is going to celebrate the beginning of the year of the dragon. He lives close to two pubs. The probability that he would go to pub A is 0.5 and the probability that the would go to pub B is 0.4. In addition, the probability that he would go to at least one of the two venues is 0.8.

1. What is the sample space (in terms of A and B)?
2. What is the probability that he would go to both pubs?
3. What is the probability that he would go to exactly one pub?

Solution:

1. Let A be the event that he would go to pub A, and let B be the event that he goes to pub B. Therefore,

$$S = \left\{A \cap B^{\mathrm{C}}, A^{\mathrm{C}} \cap B, A \cap B, A^{\mathrm{C}} \cap B^{\mathrm{C}}\right\}$$

2. The probability that he would go to both pubs is

$$\begin{aligned} \mathrm{P}(A \cap B) &= \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cup B) \\ &= 0.5 + 0.4 - 0.8 \\ &= 0.1 \end{aligned}$$

3. The probability that he would go to exactly one pub is

$$\begin{aligned} \mathrm{P}(A \cup B) - \mathrm{P}(A \cap B) &= 0.8 - 0.1 \\ &= 0.7 \end{aligned}$$

$\square$

Example 33. — **A fair coin is flipped 3 times** . What is the probability of obtaining heads on the first flip OR the third flip?

Solution: the sample space is given by

$$\begin{array}{ccc} H & H & H \\ H & H & T \\ H & T & H \\ H & T & T \\ T & H & H \\ T & H & T \\ T & T & H \\ T & T & T \end{array}$$

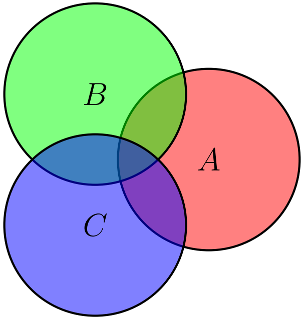
Let H_1 and H_3 denote the events that the first flip results in heads and the third flip results in heads, respectively. By the inclusion-exclusion formula, we have

$$\begin{aligned} \mathrm{P}(H_1 \cup H_3] &= \mathrm{P}(H_1] + \mathrm{P}(H_3] - \mathrm{P}(H_1 \cap H_3] \\ &= \frac{1}{2} + \frac{1}{2} - \frac{1}{4} \\ &= \frac{3}{4} \end{aligned}$$

$\square$

Fact 1.16 — Addition rule for 3 events.

$$\mathrm{P}(A \cup B \cup C) = \mathrm{P}(A) + \mathrm{P}(B) + \mathrm{P}(C) - (\mathrm{P}(A \cap B) + \mathrm{P}(A \cap C) + \mathrm{P}(B \cap C)) + \\ + \mathrm{P}(A \cap B \cap C)$$



Fact 1.17 — Addition rule for n events.

$$
\begin{aligned}
\mathrm{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathrm{P}(A_i) &- \sum \mathrm{P}(\text{all 2-event intersections}) \\
&+ \sum \mathrm{P}(\text{all 3-event intersections}) \\
&- \sum \mathrm{P}(\text{all 4-event intersections}) \\
&\dots
\end{aligned}
$$

Fact 1.18 — Addition rule for n events using DeMorgan's law.

$$\mathrm{P}\left(\bigcup_{i=1}^{n} A_i\right) = 1 - \mathrm{P}(\bigcap_{i=1}^{n} {A_i}^c)$$

For example,

$$\mathrm{P}(A \cup B \cup C) = 1 - \mathrm{P}(A^c \cap B^c \cap C^c)$$

Example 34. — The weatherman said that:

$$\mathrm{P}(\text{rain Mon}) = 30\%, \qquad \mathrm{P}(\text{rain Tue}) = 40\%, \qquad \mathrm{P}(\text{rain Wed}) = 50\%.$$

From experience, we know that

$$\mathrm{P}(\text{rain 2 days in a row}) = 20\%, \qquad \mathrm{P}(\text{rain 3 days in a row}) = 10\%,$$

$$\mathrm{P}(\text{rain Mon, Wed}) = 5\%.$$

Show that there is an 85% percent chance of rain anytime from Monday to Wednesday.

Solution: Let A, B and C be the events that it rains Monday, Tuesday and Wednesday, respectively. Then,

$$
\begin{aligned}
\mathrm{P}(A \cup B \cup C) &= \mathrm{P}(A) + \mathrm{P}(B) + \mathrm{P}(C) - (\mathrm{P}(A \cap B) + \mathrm{P}(A \cap C) + \mathrm{P}(B \cap C)) + \\
&+ \mathrm{P}(A \cap B \cap C) \\
&= .3 + .4 + .5 - (.2 + .05 + .2) + .1 = .85
\end{aligned}
$$

$\square$

Example 35. — 3 dice are rolled. What is the probability that (at least) one of the dice results in 4?

Solution:[1] Let $F_i, i \in \{1,2,3\}$ be the event that the ith dice results in a 4. We are interested in $\mathrm{P}(F_1 \cup F_2 \cup F_3]$. By the inclusion-exclusion formula we have

$$\mathrm{P}(F_1 \cup F_2 \cup F_3) = \mathrm{P}(F_1) + \mathrm{P}(F_2) + \mathrm{P}(F_3) - \mathrm{P}(F_1 \cap F_2) - \mathrm{P}(F1 \cap F_3) - \mathrm{P}(F_2 \cap F_3) + \mathrm{P}(F_1 \cap F_2 \cap F_3)$$

Using combinatorics ($\rightarrow$ see spreadsheet):

$$\begin{aligned} |F_i| &= 6^2, i = 1,2,3 \\ |F_i \cap F_j| &= 6, j \neq i = 1,2,3 \\ |F_1 \cap F_2 \cap F_3| &= 1 \end{aligned}$$

and thus

$$\begin{aligned} |F_1 \cup F_2 \cup F_3| &= |F_1| + |F_2| + |F_3| - |F_1 \cap F_2| - |F1 \cap F_3| - |F_2 \cap F_3| + |F_1 \cap F_2 \cap F_3| \\ &= 6^2 + 6^2 + 6^2 - 6 - 6 - 6 + 1 = 91 \end{aligned}$$

Since $|S| = 6^3$, the requested probability is $91/216$.

$\square$

Solution:[2, using De Morgan]

$$\mathrm{P}(F_1 \cup F_2 \cup F_3) = 1 - \mathrm{P}({F_1}^c \cap {F_2}^c \cap {F_3}^c)$$

Using combinatorics, $|{F_1}^c \cap {F_2}^c \cap {F_3}^c| = 5^3$ because none of the three dice should land on 4. Therefore, $\mathrm{P}({F_1}^c \cap {F_2}^c \cap {F_3}^c] = 5^3/6^3$ and the requested probability is $1 - 5^3/6^3 = 91/216$.

$\square$

1.6 Conditional Probability

For two events A and B in sample space S, where $\mathrm{P}(B) > 0$, the conditional probability, i.e. the probability that A will occur after B has already occurred is defined as

$$\mathrm{P}(A|B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$$

Key idea: we divide by $\mathrm{P}(B)$ to account for the new sample space, B.

Conditional Probabilities are just like the ordinary probabilities, only on a new sample space. Thus, they satisfy all the formulas we already know, e.g.:

- $\mathrm{P}(A \cup B|C) = \mathrm{P}(A|C) + \mathrm{P}(B|C) - \mathrm{P}(A \cap B|C)$
- $\mathrm{P}(A^c|B) = 1 - \mathrm{P}(A|B)$
- $\mathrm{P}(F_1 \cup F_2 \cup F_3|C) = 1 - \mathrm{P}({F_1}^c \cap {F_2}^c \cap {F_3}^c|C)$

Example 36. A dice is rolled once. Consider the following events.

A: The result is even.

B: The result is greater than 3.

What is the probability that the result is even, if it is known that result is greater than 3?

Solution:

$$
\begin{aligned}
\mathrm{P}(A|B) &= \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} \\
&= \frac{\mathrm{P}\left(\{2,4,6\} \cap \{4,5,6\}\right)}{\mathrm{P}\left(\{4,5,6\}\right)} \\
&= \frac{\mathrm{P}\left(\{4,6\}\right)}{\mathrm{P}\left(\{4,5,6\}\right)} \\
&= \frac{1/3}{1/2} \\
&= \frac{2}{3}
\end{aligned}
$$

$\square$

Example 37. A coin is flipped twice. What is the probability of getting 'Heads' on both flips, given that the first flip results in 'Heads'.

Solution:

$$S = \{(H,H),(H,T),(T,H),(T,T)\}$$

Let A be the event of getting two 'Heads'. Therefore,

$$A = \{(H,H)\}$$

Let B be the event that the first flip results in 'Heads'. Therefore,

$$B = \{(H,T),(H,H)\}$$

Therefore,

$$\begin{aligned}
\mathrm{P}(A|B) &= \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} \\
&= \frac{1/4}{1/2} \\
&= \frac{1}{2}
\end{aligned}$$

$\square$

Example 38. A coin is flipped twice. What is the probability of 'Heads' on both flips, given that at least one flip results in 'Heads'.

Solution:

$$S = \{(H,H),(H,T),(T,H),(T,T)\}$$

Let A be the event of getting two 'Heads'. Therefore,

$$A = \{(H,H)\}$$

Let B be the event that at least one flip results in 'Heads'. Therefore,

$$B = \{(H,T),(T,H),(H,H)\}$$

Therefore,

$$\begin{aligned}
\mathrm{P}(A|B) &= \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} \\
&= \frac{1/4}{3/4} \\
&= \frac{1}{3}
\end{aligned}$$

□

Example 39. The probability that a new car battery functions for over 10,000 miles is 0.8, the probability that it functions for over 20,000 miles is 0.4, and the probability that it functions for over 30,000 miles is 0.1. If a new car battery is still working after 10,000 miles, what is the probability that

a) its total life will exceed 20,000 miles,
b) its additional life will exceed 20,000 miles?

Consider the following events to answer the questions:

L_{10}: event that the battery lasts for more than 10K miles.
L_{20}: event that the battery lasts for more than 20K miles.
L_{30}: event that the battery lasts for more than 30K miles.

Solution:
We know that $\mathrm{P}(L_{10}) = 0.8$, $\mathrm{P}(L_{20}) = 0.4$ and $\mathrm{P}(L_{30}) = 0.1$. We are interested in calculating $\mathrm{P}(L_{20}|L_{10})$ and $\mathrm{P}(L_{30}|L_{10})$.

$$\begin{aligned}
\mathrm{P}(L_{20}|L_{10}) &= \frac{\mathrm{P}(L_{20} \cap L_{10})}{\mathrm{P}(L_{10})} \\
&= \frac{\mathrm{P}(L_{20})}{\mathrm{P}(L_{10})} \\
&= \frac{0.4}{0.8} \\
&= \frac{1}{2}
\end{aligned}$$

By doing similar calculations it is easy to verify that $\mathrm{P}(L_{30}|L_{10}) = \frac{1}{8}$. $\square$

Fact 1.19 — Multiplication Rule. For two events:

$$
\begin{aligned}
P(A_1 \cap A_2) &= P(A_2)P(A_1|A_2) \\
&= P(A_1)P(A_2|A_1)
\end{aligned}
$$

For 3 events there are 3! possibilities:

$$
\begin{aligned}
P(A_1 \cap A_2 \cap A_3) &= P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) \\
&= P(A_2)P(A_3|A_2)P(A_1|A_2 \cap A_3) \\
&= \dots
\end{aligned}
$$

For 4 events we have 4! alternative formulas:

$$
\begin{aligned}
P(A_1 \cap A_2 \cap A_3 \cap A_4) &= P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2)P(A_4|A_1 \cap A_2 \cap A_3) \\
&= P(A_4)P(A_2|A_4)P(A_3|A_4 \cap A_2)P(A_1|A_4 \cap A_2 \cap A_3) \\
&= \dots
\end{aligned}
$$

Example 40. An urn initially contains 5 white balls and 7 black balls. Each time a ball is selected, its color is noted and it is **replaced in the urn along with two other balls of the same color**. Compute the probability that the first two balls selected are black and the next two white. Consider the following events to answer the question:

B_1: event that the first ball chosen is black.
B_2: event that the second ball chosen is black.
W_3: event that the third ball chosen is white.
W_4: event that the fourth ball chosen is white.

Solution: We are interested in calculating $\mathrm{P}(B_1 \cap B_2 \cap W_3 \cap W_4)$. Using the Multiplication rule we get,

$$\begin{aligned}
\mathrm{P}(B_1 \cap B_2 \cap W_3 \cap W_4) &= \mathrm{P}(B_1) \cdot \mathrm{P}(B_2|B_1) \cdot \mathrm{P}(W_3|B_1 \cap B_2) \cdot \mathrm{P}(W_4|B_1 \cap B_2 \cap W_3) \\
&= \frac{7}{12} \times \frac{9}{14} \times \frac{5}{16} \times \frac{7}{18} \\
&= \frac{35}{768}
\end{aligned}$$

$\square$

Example 41. A deck of cards is randomly divided into four stacks of 13 cards each. Find the probability that each stack has exactly one ace. Hint:

Let A_1 be the event that A♠ is in any one of the stacks.
Let A_2 be the event that A♠, A♡ are in different stacks.
Let A_3 be the event that A♠, A♡, A♢ are in different stacks.
Let A_4 be the event that A♠, A♡, A♢, A♣ are in different stacks.

Solution: [using conditional probability.]
Let A_1 be the event that A♠ is in any one of the stacks.
Let A_2 be the event that A♠, A♡ are in different stacks.
Let A_3 be the event that A♠, A♡, A♢ are in different stacks.
Let A_4 be the event that A♠, A♡, A♢, A♣ are in different stacks.
Therefore,

$$\begin{aligned}\mathrm{P}(A_1 \cap A_2 \cap A_3 \cap A_4) &= \mathrm{P}(A_1)\mathrm{P}(A_2|A_1)\mathrm{P}(A_3|A_1 \cap A_2)\mathrm{P}(A_4|A_1 \cap A_2 \cap A_3) \\ &= 1 \quad \times \quad \frac{39}{51} \quad \times \quad \frac{26}{50} \quad \times \quad \frac{13}{49} \quad = 0.105\end{aligned}$$

□

Solution: [using combinatorics.]

$$|S| = \binom{52}{13}\binom{39}{13}\binom{26}{13}\binom{13}{13}$$

Let A be the event that each stack has exactly one ace. Therefore, each stack has one ace, and 12 non-ace cards. Therefore,

$$|A| = \left(\binom{4}{1}\binom{48}{12}\right)\left(\binom{3}{1}\binom{36}{12}\right)\left(\binom{2}{1}\binom{24}{12}\right)\left(\binom{1}{1}\binom{12}{12}\right)$$

Therefore,

$$\begin{aligned} \mathrm{P}(A) &= \frac{|A|}{|S|} \\ &= \frac{\left(\binom{4}{1}\binom{48}{12}\right)\left(\binom{3}{1}\binom{36}{12}\right)\left(\binom{2}{1}\binom{24}{12}\right)\left(\binom{1}{1}\binom{12}{12}\right)}{\binom{52}{13}\binom{39}{13}\binom{26}{13}\binom{13}{13}} &= 0.105 \end{aligned}$$

$\square$

Example 42. (Adopted from Meyer) A 52-card deck is thoroughly shuffled and you are dealt a hand of 13 cards.

1. If you have (at least) one ace, what is the probability that you have a second ace?
2. If you have (at least) the ace of spades, what is the probability that you have a second ace?

Hint: Define the events:
A1: exactly one ace
A2: exactly two aces
A3: exactly three aces
A4: exactly four aces

Solution: Define the events:
A1: exactly one ace, A2: exactly two aces, A3: exactly three aces, A4: exactly four aces.

Note that these events are disjoint (= Mutually exclusive). Remember that if A and B are two disjoint events, $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B)$.

a) The problem is asking for the probability: $\mathrm{P}(A2 \cup A3 \cup A4 | A1 \cup A2 \cup A3 \cup A4)$

Compute the probabilities of each of the four events:

$\mathrm{P}(A1) = \frac{\binom{4}{1}\binom{48}{12}}{\binom{52}{13}} = 0.438 \quad \mathrm{P}(A2) = \frac{\binom{4}{2}\binom{48}{11}}{\binom{52}{13}} = 0.213 \quad \mathrm{P}(A3) = \frac{\binom{4}{3}\binom{48}{10}}{\binom{52}{13}} = 0.0412 \quad \mathrm{P}(A4) = \frac{\binom{4}{4}\binom{48}{9}}{\binom{52}{13}} = 0.00264$

From the definition of conditional probability:

$$\begin{aligned}
\mathrm{P}(A2 \cup A3 \cup A4 | A1 \cup A2 \cup A3 \cup A4) &= \frac{\mathrm{P}((A2 \cup A3 \cup A4) \cap (A1 \cup A2 \cup A3 \cup A4))}{\mathrm{P}(A1 \cup A2 \cup A3 \cup A4)} \\
&= \frac{\mathrm{P}(A2 \cup A3 \cup A4)}{\mathrm{P}(A1 \cup A2 \cup A3 \cup A4)} \\
&= \frac{\mathrm{P}(A2) + \mathrm{P}(A3) + \mathrm{P}(A4)}{\mathrm{P}(A1) + \mathrm{P}(A2) + \mathrm{P}(A3) + \mathrm{P}(A4)} \\
&\approx \frac{0.257}{0.7} = 0.37
\end{aligned}$$

b) Remarkably, the answer is different from part (a).

$\mathrm{P}(A1) = \frac{\binom{1}{1}\binom{48}{12}}{\binom{52}{13}} = 0.109$ $\quad \mathrm{P}(A2) = \frac{\binom{1}{1}\binom{3}{1}\binom{48}{11}}{\binom{52}{13}} = 0.106$ $\quad \mathrm{P}(A3) = \frac{\binom{1}{1}\binom{3}{2}\binom{48}{10}}{\binom{52}{13}} = 0.0309$ $\quad \mathrm{P}(A4) = \frac{\binom{1}{1}\binom{3}{3}\binom{48}{9}}{\binom{52}{13}} = 0.00264$

$\mathrm{P}(A2 \cup A3 \cup A4 | A1 \cup A2 \cup A3 \cup A4) \approx \frac{0.139}{0.249} = 0.56$

$\square$

Example 43. What is the probability that when a deck of cards is dealt in a game of bridge (each player gets 13 cards), the $\heartsuit$s will be dealt such that Alice gets 3, Bob gets 4, Charlie gets 2, David gets 4.

Solution: Let E_{Alice} be the event of Alice getting 3 $\heartsuit$s.
Let E_{Bob} be the event of Bob getting 4 $\heartsuit$s.
Let E_{Charlie} be the event of Charlie getting 2 $\heartsuit$s.
Let E_{David} be the event of David getting 4 $\heartsuit$s.
Therefore,

$$\mathrm{P}(E_{\text{Alice}}) = \frac{\binom{13}{3}\binom{39}{10}}{\binom{52}{13}}$$

$$\mathrm{P}(E_{\text{Bob}}|E_{\text{Alice}}) = \frac{\binom{10}{4}\binom{29}{9}}{\binom{39}{13}}$$

$$\mathrm{P}(E_{\text{Charlie}}|E_{\text{Alice}} \cap E_{\text{Bob}}) = \frac{\binom{6}{2}\binom{20}{11}}{\binom{26}{13}}$$

$$\mathrm{P}(E_{\text{David}}|E_{\text{Alice}} \cap E_{\text{Bob}} \cap E_{\text{Charlie}}) = \frac{\binom{4}{4}\binom{9}{9}}{\binom{13}{13}}$$

Therefore,

$$\begin{aligned}
\mathrm{P}(E_{\text{Alice}} \cap E_{\text{Bob}} \cap E_{\text{Charlie}} \cap E_{\text{David}}) = &\quad \mathrm{P}(E_{\text{Alice}}) \\
&\times \mathrm{P}(E_{\text{Bob}} | E_{\text{Alice}}) \\
&\times \mathrm{P}(E_{\text{Charlie}} | E_{\text{Alice}} \cap E_{\text{Bob}}) \\
&\times \mathrm{P}(E_{\text{David}} | E_{\text{Alice}} \cap E_{\text{Bob}} \cap E_{\text{Charlie}}) \\
= &\frac{\binom{13}{3}\binom{39}{10}}{\binom{52}{13}} \frac{\binom{10}{4}\binom{29}{9}}{\binom{39}{13}} \frac{\binom{6}{2}\binom{20}{11}}{\binom{26}{13}} \frac{\binom{4}{4}\binom{9}{9}}{\binom{13}{13}}
\end{aligned}$$

$\square$

1.7 Independent Events

Two events, A and B, are said to be independent if

$$P(A|B) = P(A)$$

$\rightarrow$ Information about the occurrence of B does not affect the probability of A.

Fact 1.20 — The multiplication rule with independent events.

$$P(A \cap B) = P(A)P(B)$$

if and only if A and B are independent.

Fact 1.21 If A and B are independent, then so are A^c and B, A and B^c, A^c and B^c

Three events, A, B, and C, are said to be:

mutually independent if

$$P(A \cap B \cap C) = P(A)P(B)P(C)$$

and *pairwise* independent if

$$P(A \cap B) = P(A)P(B)$$
$$P(B \cap C) = P(B)P(C)$$
$$P(C \cap A) = P(C)P(A)$$

STOP! Pairwise independence does not imply mutual independence!

Example 44. two fair coins are tossed. Let
A: the first coin is H
B: the second coin is H
C: both coins match

a) are they pairwise independence?
b) are they mutually independent?

Solution:

$$
\begin{aligned}
S &= \{(H,H),(H,T),(T,H),(T,T)\} \\
A &= \{(H,H),(H,T)\} \qquad B = \{(H,H),(T,H)\} \qquad C = \{(H,H),(T,T)\} \\
A\cap B &= \{(H,H)\} \qquad A\cap C = \{(H,H)\} \qquad B\cap C = \{(H,H)\} \\
A\cap B\cap C &= \{(H,H)\}
\end{aligned}
$$

a) they are pairwise independent because

$$
\begin{aligned}
\mathrm{P}(A\cap B) &= \mathrm{P}(A)\mathrm{P}(B) = 1/4 \\
\mathrm{P}(B\cap C) &= \mathrm{P}(B)\mathrm{P}(C) = 1/4 \\
\mathrm{P}(C\cap A) &= \mathrm{P}(C)\mathrm{P}(A) = 1/4
\end{aligned}
$$

b) they are not mutually independent because

$$\mathrm{P}(A\cap B\cap C) = 1/4 \neq \mathrm{P}(A)\mathrm{P}(B)\mathrm{P}(C) = (1/2)^3$$

This makes sense: knowing that B and C occurred tells us that A also did. □

Example 45. — **3 dice are rolled,** what is the probability that one of the dice results in 4?

Solution:[1] Let $F_i, i \in \{1,2,3\}$ be the event that the ith dice results in a 4. We are interested in $\mathrm{P}(F_1 \cup F_2 \cup F_3)$. By the inclusion-exclusion formula we have $\mathrm{P}(F_1 \cup F_2 \cup F_3) = \mathrm{P}(F_1) + \mathrm{P}(F_2) + \mathrm{P}(F_3) - \mathrm{P}(F_1 \cap F_2) - \mathrm{P}(F1 \cap F_3) - \mathrm{P}(F_2 \cap F_3) + \mathrm{P}(F_1 \cap F_2 \cap F_3)$ Since the events F_1, F_2, F_3 are mutually independent we can rewrite the above expression as

$$\begin{aligned}
\mathrm{P}(F_{\cup} F_2 \cup F_3) &= \mathrm{P}(F_1) + \mathrm{P}(F_2) + \mathrm{P}(F_3) - \mathrm{P}(F_1)\mathrm{P}(F_2) - \mathrm{P}(F1)\mathrm{P}(F_3) - \mathrm{P}(F_2)\mathrm{P}(F_3) \\
&\quad + \mathrm{P}(F_1)\mathrm{P}(F_2)\mathrm{P}(F_3) \\
&= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} - \left(\frac{1}{6} \times \frac{1}{6}\right) - \left(\frac{1}{6} \times \frac{1}{6}\right) - \left(\frac{1}{6} \times \frac{1}{6}\right) + \left(\frac{1}{6} \times \frac{1}{6} \times \frac{1}{6}\right) \\
&= \frac{91}{216}
\end{aligned}$$

□

Solution:[2, using De Morgan]

$$\mathrm{P}(F_1 \cup F_2 \cup F_3) = 1 - \mathrm{P}({F_1}^c \cap {F_2}^c \cap {F_3}^c) = 1 - (5/6)^3 = \frac{91}{216}$$

$\square$

Example 46. A biased coin with $p =$ probability of coming up H, is tossed n times. What is the probability of having at least one H ?

Solution: Let A = having at least one H out of the n tosses. Instead of enumerating all possible events containing at least one H and then compute the union of all those events, it is easier to note that A^c = having all tosses come up T. Since $\mathrm{P}(T) = 1 - p$ and all the tosses are independent, $\mathrm{P}(A^c) = (1-p)^n$ and the desired probability is

$$\boxed{\mathrm{P}(A) = 1 - (1-p)^n} \tag{1.4}$$

STOP! Note that combinatorics is not useful here because the coin is biased, which means that all outcomes in the relevant sample space are not equally likely.

If the coin were fair, i.e. $p = 1/2$, we could use combinatorics and get $|S| = 2^n, |A^c| = 1$ and $\mathrm{P}(A^c) = 1/2^n$, which matches the more general results above that $\mathrm{P}(A^c) = (1-p)^n = (1-1/2)^n$.

$\square$

Example 47. Suppose that A, B are mutually exclusive and $\mathrm{P}(A) > 0$ and $\mathrm{P}(B) > 0$. Are they independent?

Solution: No! Since $\mathrm{P}(A \cap B) = 0$ for mutuality exclusive events, knowing that one occurred means that the other cannot. Mathematically, they do not satisfy the condition for the independence

$$\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B)$$

in this case (where $\mathrm{P}(A) > 0$ and $\mathrm{P}(B) > 0$). $\square$

Example 48. Suppose that $A \subset B$ and $\mathrm{P}(A) > 0$ and $\mathrm{P}(B) > 0$. Are two events A and B independent?

Solution: Since $A \subset B$,

$$\mathrm{P}(A \cap B) = \mathrm{P}(A)$$

The condition for the independence is

$$\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B)$$

Hence, if $\mathrm{P}(B) = 1$, A and B are independent but if $\mathrm{P}(B) < 1$, A and B are not independent.

$\square$

Example 49. If A and B are independent events, with $\mathrm{P}(A) = \frac{1}{3}$ and $\mathrm{P}(B) = \frac{1}{4}$, find the following:
(a) $\mathrm{P}(A^c \cap B^c)$
(b) $\mathrm{P}(A^c|B)$.

Solution: a) Since A and B are independent, A^c and B^c are independent. So $\mathrm{P}(A^c \cap B^c) = \mathrm{P}(A^c)\mathrm{P}(B^c) = \Big(1 - \mathrm{P}(A)\Big)\Big(1 - \mathrm{P}(B)\Big) = (1 - \frac{1}{3})(1 - \frac{1}{4}) = \frac{1}{2}$.

b) Since A and B are independent, A^c and B are also independent. So $\mathrm{P}(A^c|B) = \mathrm{P}(A^c) = 1 - \mathrm{P}(A) = \frac{2}{3}$.

$\square$

Example 50. Two fair dice are rolled.
Let A be the event that the sum of the results of the dice is 6.
Let B be the event that the result of the first dice is 4.
Let C be the event that the sum of the results of the dice is 7.
Which of the possible pairs of the events are independent?

Solution:
Let A be the event that the sum of the results of the dice is 6.
Let B be the event that the result of the first dice is 4.
Let C be the event that the sum of the results of the dice is 7.

$$\mathrm{P}(A) = \frac{5}{36} \qquad \mathrm{P}(B) = \frac{1}{6} \qquad \mathrm{P}(C) = \frac{6}{36}$$
$$\mathrm{P}(A \cap B) = \frac{1}{36} \qquad \mathrm{P}(B \cap C) = \frac{1}{36} \qquad \mathrm{P}(A \cap C) = 0$$

Therefore,

$$\begin{aligned} \mathrm{P}(A \cap B) &\neq \mathrm{P}(A)\mathrm{P}(B) \\ \mathrm{P}(B \cap C) &= \mathrm{P}(B)\mathrm{P}(C) \\ \mathrm{P}(C \cap A) &\neq \mathrm{P}(C)\mathrm{P}(A) \end{aligned}$$

Therefore, only B and C are independent. Why? Think about the meaning of $\mathrm{P}(B|A)$ and $\mathrm{P}(B|C)$.
$\square$

Example 51. Two cards are sequentially drawn (without replacement) from a well-shuffled deck of 52 cards. Let A be the event that the two cards drawn have the same value (e.g. both 4s) and let B be the event that the first card drawn is an ace. Are these events independent?

Solution: Recall: Let A be the event that the two cards drawn have the same value (e.g. both 4s) and let B be the event that the first card drawn is an ace. To decide whether the two events are independent we need to check whether $\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B)$.

$$\begin{aligned}
\mathrm{P}(A) &= \frac{52 \times 3}{52 \times 51} = \frac{1}{17} \\
\mathrm{P}(B) &= \frac{4 \times 51}{52 \times 51} = \frac{1}{13} \\
\mathrm{P}(A \cap B) &= \frac{4 \times 3}{52 \times 51} \\
&= \frac{1}{17} \times \frac{1}{13} \\
&= \mathrm{P}(A)\mathrm{P}(B)
\end{aligned}$$

So yes, they are independent! This makes sense, all pairs have the same probability of being dealt.
$\square$

1.8 Law of Total Probability

Fact 1.22 — Law of Total Probability . Given a division $A_1, \ldots, A_n$ of the sample space S, and an event B in S,

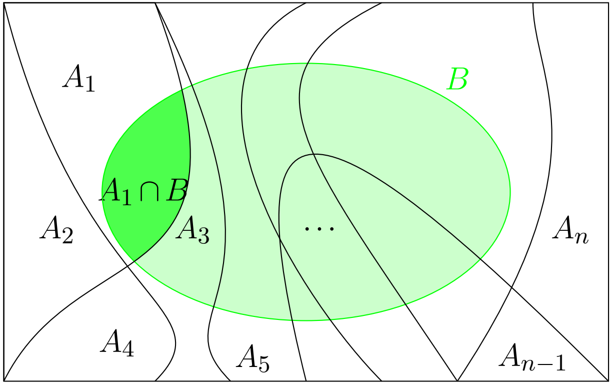
$$P(B) = \sum_{i=1}^{n} P(B|A_i)P(A_i)$$

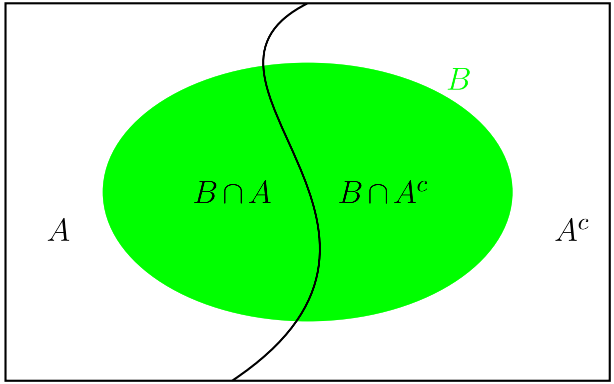
For example, for $n = 3$:

$$P(B) = P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + P(B|A_3)P(A_3).$$

For $n = 2$, we can say $A_1 = A$ and $A_2 = A^c$, so:

$$\boxed{P(B) = P(B|A)P(A) + P(B|A^c)P(A^c).} \tag{1.5}$$


A_1
B
$A_1 \cap B$
A_2
A_3
$\cdots$
A_n
A_4
A_5
A_{n-1}


B
$B \cap A$
$B \cap A^c$
A
A^c

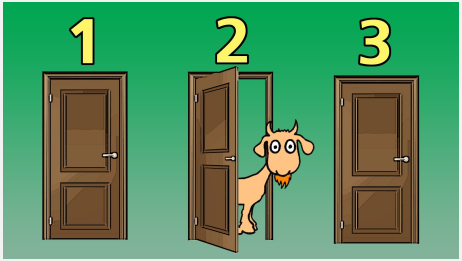
Example 52. * A chocolate factory has three production lines.
50% of the production is milk chocolate, out of which 1% is defective.
30% of the production is dark chocolate, out of which 2% is defective.
20% of the production is white chocolate, out of which 0.5% is defective.
If a chocolate bar is picked randomly, what is the probability that it is defective?
Hint: Let A_1, A_2, A_3 be the events that selected chocolate bar is made of milk, dark, white chocolate, respectively. Let B be the event that the selected chocolate bar is defective.

Solution: Therefore,

$$\begin{aligned}
\mathrm{P}(B) &= \sum_{i=1}^{3} \mathrm{P}(A_i)\mathrm{P}(B|A_i) \\
&= ((0.5)(0.01)) + ((0.3)(0.02)) + ((0.2)(0.005)) \\
&= 0.005 + 0.006 + 0.001 \\
&= 0.012
\end{aligned}$$

$\square$

Example 53. — Monty Hall Problem. You're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 2, which has a goat. He then says to you, "Do you want to pick door No. 3?" Is it to your advantage to switch your choice?



Hint: use the total probability rule with the events:

W: win by switching doors
E: car behind original door (before Monty Hall opens door No. 2 above)

Solution: Let:
W: win by switching doors
E: car behind original door (before Monty Hall opens door No. 2 above)

$$\begin{aligned} P(W) &= P(W|E)P(E) + P(W|E^c)P(E^c) \\ &= (0)(1/3) + (1)(2/3) \\ &= 2/3 \end{aligned}$$

$\square$

Example 54. — Tornadoes** A structure is located in a region where tornado wind force must be considered in its design. Suppose that from the records of tornadoes for the past 200 years, it is estimated that during any 5-year period the probability of having 0, 1 and 2 tornadoes is 0.5, 0.3 and 0.2, respectively. If a tornado occurs, the structure will be damaged with probability $p = 5\%$.

a) if two tornadoes occurred last year, what is the probability that there was damage to the structure?

b) what is the probability the a structure will be damaged in the next five years?

Hint: For part a) let

A_i be the events of having $i = 0, 1, 2$ tornadoes last year, and
D the event that a particular structure was damaged last year.

Solution:
...it is estimated that during any 5-year period the probability of having 0, 1 and 2 tornadoes is 0.5, 0.3 and 0.2, respectively. If a tornado occurs, the structure will be damaged with probability $p = 5\%$.

a) If two tornadoes occurred last year, what is the probability that there was damage to the structure? Let:

$$A_i = \text{having } i = 0, 1, 2 \text{ tornadoes last year, and}$$

$$D = \text{a structure was damaged last year.}$$

For any particular structure, it is easier to calculate the probability of no damage given the two tornadoes $(= (1-p)^2)$, so

$$\begin{aligned} \mathrm{P}(D|A_2) &= 1 - \mathrm{P}(D^c|A_2) \\ &= 1 - (1-p)^2 = 0.0975 \end{aligned}$$

b) what is the probability that the structure will be damaged in the next five years? Let

A_i = having $i = 0,1,2$ tornadoes in the next five years, and

D = the structure will be damaged in the next five years.
Since we don't know the number of tornadoes that will occur, we use the total probability rule:

$$\begin{aligned}
\mathrm{P}(D^c) &= \sum_{i=0}^{2} \mathrm{P}(D^c|A_i)\mathrm{P}(A_i) \\
&= \sum_{i=0}^{2} (1-p)^i \mathrm{P}(A_i) \\
&= (1-p)^0\mathrm{P}(A_0) + (1-p)^1\mathrm{P}(A_1) + (1-p)^2\mathrm{P}(A_2) \\
&= ((1)(0.5)) + ((0.95)(0.3)) + \big((0.95^2)(0.2)\big) = 0.9655
\end{aligned}$$

and the answer is 1-0.9655=0.0345.

$\square$

Example 55. A student in Monty Hall's probability course misses his exam and must take a makeup. Before the test, Prof. Hall invites him to choose one of five envelopes, two containing easy makeup exams, three containing hard ones, and the student takes an envelope.
"Before you start, you might enjoy looking at one of my hard makeup exams – just full of nasty probability problems," says the professor. From among the four remaining envelopes, he selects one at random that he knows to contain a hard exam and opens it.
"Excuse me," says the student, "but the envelope I picked looks a bit smudged. Could I swap it for one of the others?" And he does.
What is the probability that the student has an easy exam after making the swap?

Solution: Let:
Event W: easy exam after swap
Event E: first choice is an easy exam
Event H: first choice is a hard exam
$\mathrm{P}(W) = \mathrm{P}(W|E)\mathrm{P}(E) + \mathrm{P}(W|H)\mathrm{P}(H) = (1/3)(2/5) + (2/3)(3/5) = 8/15$

$\square$

Example 56. A box contains w white balls, b black balls and r red balls. A ball is chosen at random and if it is either black or red then it is replaced by a white ball and if it is white then it is replaced by a red ball. Now again draw a ball.

a) What is the probability that the second ball drawn is red when the first ball drawn is red ?

b) What is the probability that the second ball drawn is white?

Solution: Let W_i, B_i, R_i be the event that the $i-$th draw is a white, black and red ball respectively.
a)

$$\mathrm{P}(R_2|R_1) = \frac{r-1}{w+b+r}.$$

b)

$$\begin{aligned}
\mathrm{P}(W_2) = &\quad \mathrm{P}(W_2|W_1)\mathrm{P}(W_1) + \mathrm{P}(W_2|B_1)\mathrm{P}(B_1) + \mathrm{P}(W_2|R_1)\mathrm{P}(R_1) \\
= &\quad \frac{w-1}{w+b+r}\frac{w}{w+b+r} + \frac{w+1}{w+b+r}\frac{b}{w+b+r} + \frac{w+1}{w+b+r}\frac{r}{w+b+r}.
\end{aligned}$$

$\square$

1.9 Bayes' Theorem

Bayes rule with two events

$$
\begin{aligned}
\mathrm{P}(A|B) &= \frac{\mathrm{P}(A\cap B)}{\mathrm{P}(B)} && \text{(conditional probability)} \\
&= \frac{\mathrm{P}(B|A)\mathrm{P}(A)}{\mathrm{P}(B)} && \text{(Bayes rule v1)} \\
&= \frac{\mathrm{P}(B|A)\mathrm{P}(A)}{\mathrm{P}(B|A)\mathrm{P}(A)+\mathrm{P}(B|A^c)\mathrm{P}(A^c)} && \text{(Bayes rule v2)}
\end{aligned}
$$

Bayes rule with a division of S. If we have a division $A_1, \ldots, A_n$ of the sample space S, then by the Law of Total Probability $\mathrm{P}(B) = \sum_{i=1}^{n} \mathrm{P}(B|A_i)\mathrm{P}(A_i)$ and:

$$\begin{aligned} \mathrm{P}(A_i|B) &= \frac{\mathrm{P}(A_i \cap B)}{\mathrm{P}(B)} && \text{(conditional probability)} \\ &= \frac{\mathrm{P}(B|A_i)\mathrm{P}(A_i)}{\sum_{j=1}^{n} \mathrm{P}(B|A_j)\mathrm{P}(A_j)} && \text{(Bayes rule v3)} \end{aligned}$$

Example 57. In a bolt factory, machines 1,2 and 3 respectively produce 20 %, 30% and 50% of the total output. Of their output, 5%, 3% and 2% are defective. A bolt is selected at random.

a) What is the probability that it is defective?

b) Given that it is defective, what is the probability that it was made by machine 1?

Hint: Let D be the event that the bolt is defective and M_1, M_2, M_3 be the events that the selected bolt comes from machines 1,2 and 3 respectively.

Solution: We have
$P(M_1) = 0.2, P(M_2) = 0.3, P(M_3) = 0.5$ and
$P(D|M_1) = 0.05, P(D|M_2) = 0.03, P(D|M_3) = 0.02.$
a) From the law of total probability,

$$P(D) = P(D|M_1)P(M_1) + P(D|M_2)P(M_2) + P(D|M_3)P(M_3) = 0.029$$

b)

$$P(M_1|D) = \frac{P(D|M_1)P(M_1)}{P(D)} = 0.52$$

□

Example 58. $1/10$ of men and $1/7$ of women are color-blind. A person is chosen at random and that person is color-blind. What is the probability that the person is male. Assume males and females to be in equal numbers. Let M=male, F=female, C=color-blind.

Solution: Let M=male, F=female, C=color-blind. Then

$$\begin{aligned}
\mathrm{P}(M|C) &= \frac{\mathrm{P}(M \cap C)}{\mathrm{P}(C)} \\
&= \frac{\mathrm{P}(C|M)\mathrm{P}(M)}{\mathrm{P}(C|M)\mathrm{P}(M) + \mathrm{P}(C|F)\mathrm{P}(F)} \\
&= \frac{\frac{1}{10} \cdot \frac{1}{2}}{\frac{1}{10} \cdot \frac{1}{2} + \frac{1}{7} \cdot \frac{1}{2}}.
\end{aligned}$$

$\square$

Example 59. A transmitter sends binary bits, 80% 0's and 20% 1's. When a 0 is sent, the receiver will detect it correctly 80% of the time. When a 1 is sent, the receiver will detect it correctly 90% of the time.
(a) What is the probability that a 1 is sent and a 1 is received?
(b) If a 1 is received, what is the probability that a 1 was sent?

Solution: We will consider the following events.

S_0: event that the transmitter sent a 0.
S_1: event that the transmitter sent a 1.
R_1: event that 1 was received.

(a) We are interested in finding $\mathrm{P}(S_1 \cap R_1)$.

$$\begin{aligned}
\mathrm{P}(S_1 \cap R_1) &= \mathrm{P}(R_1|S_1)\mathrm{P}(S_1) \\
&= 0.2 \times 0.9 \\
&= 0.18
\end{aligned}$$

(b) We are interested in finding $\mathrm{P}(S_1|R_1)$.

$$\begin{aligned}
\mathrm{P}(S_1|R_1) &= \frac{\mathrm{P}(S_1 \cap R_1)}{\mathrm{P}(R_1)} = \frac{\mathrm{P}(S_1 \cap R_1)}{\mathrm{P}(R_1 \cap S_1) + \mathrm{P}(R_1 \cap S_0)} \\
&= = \frac{\mathrm{P}(S_1 \cap R_1)}{\mathrm{P}(R_1|S_1)\mathrm{P}(S_1) + \mathrm{P}(R_1|S_0)\mathrm{P}(S_0)} = \frac{0.18}{0.18 + \mathrm{P}(R_1|S_0)\mathrm{P}(S_0)} \\
&= \frac{0.18}{0.18 + 0.8 \times 0.2} = 0.5294
\end{aligned}$$

$\square$

Example 60. An urn contains 5 white and 10 black balls. A fair die is rolled and that number of balls are chosen from the urn.
(a) What is the probability that all of the balls selected are white?
(b) What is the conditional probability that the die landed on 3 if all the balls selected are white?

Solution: We will consider the following events.

W: event that all of the balls chosen are white.
D_i: event that the die landed on i, $1 \le i \le 6$.

(a) We want to find $\mathrm{P}(W)$. We can do this as follows.

$$\begin{aligned}
\mathrm{P}(W) &= \sum_{i=1}^{6} \mathrm{P}(W \cap D_i) \\
&= \sum_{i=1}^{6} \mathrm{P}(D_i)\mathrm{P}(W|D_i) \\
&= \sum_{i=1}^{6} \frac{1}{6} \frac{\binom{5}{i}}{\binom{15}{i}} \\
&= \frac{1}{6}\left(\frac{5}{10} + \frac{10}{105} + \frac{10}{455} + \frac{5}{1365} + \frac{1}{3003}\right) \\
&= 0.1035
\end{aligned}$$

(b) We want to find $\mathrm{P}(D_3|W)$. This can be done as follows.

$$
\begin{aligned}
\mathrm{P}(D_3|W) &= \frac{D_3 \cap W}{\mathrm{P}(W)} \\
&= \frac{\mathrm{P}(D_3) \times \mathrm{P}(W|D_3)}{\mathrm{P}(W)} \\
&= \frac{1/6 \times \binom{5}{3}\binom{15}{3}}{0.1035} \\
&= \frac{1/6 \times 10/455}{0.1035} \\
&= \frac{0.00366}{0.1035} \\
&= 0.03539
\end{aligned}
$$

$\square$

Example 61. In answering a question on a multiple choice test, a student either knows the answer or the student just guesses. Suppose that the probability that the student knows the answer is 0.75. Assuming that there are 5 choices for each multiple-choice question, what is the conditional probability that the student knew the answer to a question given that the student answered it correctly?

Hint: Let:

C = student answers the question correctly,
K = student knows the answer.

Solution: Let:

C = student answers the question correctly,
K = student knows the answer.

The probability that the student who guesses will be correct is $1/5 = 0.20 = \mathrm{P}(C|K^c)$.

$$\begin{aligned}\mathrm{P}(K|C) &= \frac{\mathrm{P}(K \cap C)}{\mathrm{P}(C)} \\ &= \frac{\mathrm{P}(C|K)\mathrm{P}(K)}{\mathrm{P}(C|K)\mathrm{P}(K) + \mathrm{P}(C|K^c)\mathrm{P}(K^c)} \\ &= \frac{1 \cdot 0.75}{1 \cdot 0.75 + 0.20 \cdot 0.25} \\ &= 0.9375\end{aligned}$$

$\square$

Example 62. Stores A, B and C have 50, 75 and 100 employees respectively. and 50%, 60% and 70% of the employees are women. Resignations are equally likely among all employees, regardless of sex. One employee resigns and is a woman. What is the probability that she works in store A?

Solution: Let W be the event that a woman employee resigns from anywhere, and let A, B and C denote the event that a randomly selected employee works at the respective store. Then $\mathrm{P}(A) = 50/225$, $\mathrm{P}(B) = 75/225$ and $\mathrm{P}(C) = 100/225$. Likewise the probabilities of resignation of a woman from a store is given by the information to be $\mathrm{P}(W|A) = 0.50$, $\mathrm{P}(W|B) = 0.60$, and $\mathrm{P}(W|C) = 0.70$ Then we can use Bayes Theorem (re-deriving it in the process of using it):

$$
\begin{aligned}
\mathrm{P}(A|W) &= \frac{\mathrm{P}(A \cap W)}{\mathrm{P}(W)} \\
&= \frac{\mathrm{P}(W|C)\mathrm{P}(C)}{\mathrm{P}(W|A)\mathrm{P}(A) + \mathrm{P}(W|B)\mathrm{P}(B) + \mathrm{P}(W|C)\mathrm{P}(C)} \\
&= \frac{(0.50)(50/225)}{(0.50)(50/225) + (0.60)(75/225) + (0.70)(100/225)} \\
&\approx 0.17857
\end{aligned}
$$

□

Example 63. (Adopted from Meyer) Outside of their hum-drum duties as CS-20 Teaching Assistants, Nick is trying to learn to levitate using only intense concentration and Keenan is trying to become the world champion flaming torch juggler. Suppose that Nick's probability of success is $1/6$, Keenan's chance of success is $1/4$, and these two events are independent.

a) If at least one of them succeeds, what is the probability that Nick learns to levitate?
b) If at most one of them succeeds, what is the probability that Keenan becomes the world flaming torch juggler champion?
c) If exactly one of them succeeds, what is the probability that it is Nick?

Solution: Define the events:
N: Nick succeeds K: Keenan succeeds L: at Least one succeeds
a)

$$\begin{aligned}
\mathrm{P}(N|L) &= \frac{\mathrm{P}(L|N)\mathrm{P}(N)}{\mathrm{P}(L)} \\
&= \frac{1 \times \frac{1}{6}}{\mathrm{P}(L)} \\
&= \frac{1 \times \frac{1}{6}}{\mathrm{P}(NK^c) + \mathrm{P}(N^cK) + \mathrm{P}(NK)} \\
&= \frac{1 \times \frac{1}{6}}{\frac{3}{24} + \frac{5}{24} + \frac{1}{24}} \\
&= \frac{4}{9}
\end{aligned}$$

b) $\frac{5}{23}$
c) $\frac{3}{8}$ $\square$

1.9.1 Updating probability estimates

It is useful to use Bayes rule in terms of updating our belief about a hypothesis A in the **light of new evidence B**. We say that our **posterior** belief P(A|B) is calculated by multiplying our **prior** belief P(A) by the likelihood P(B|A) that B will occur if A is true.

The formulas are the same, e.g.

$$\mathrm{P}(A|B) = \frac{\mathrm{P}(B|A)\mathrm{P}(A)}{\mathrm{P}(B|A)\mathrm{P}(A) + \mathrm{P}(B|A^c)\mathrm{P}(A^c)}$$

but now we interpret everything on the right-hand side of this equation with the information **prior** to knowing the new evidence B.

Example 64. — Criminal investigation (adapted from Ross). In a certain stage of a criminal investigation, the inspector in charge is 60% convinced of the guilt of a certain suspect. Suppose that a new piece of DNA evidence uncovered that the criminal has diabetes. This prompted the inspector to test the suspect for diabetes, and the test came out positive, indicating that the suspect does have diabetes. If 10% of the population has diabetes how certain should that inspector be of the guilt of the suspect?
Let G be the event that the suspect is guilty, and D the event that the suspect has diabetes.

Solution:
We have $\mathrm{P}(G) = 0.6$, $\mathrm{P}(D|G) = 1$ because we know that the criminals has diabetes, $\mathrm{P}(D|G^c) = 0.1$ because **before the evidence came to light** this is our best guess. Therefore,

$$\begin{aligned}
\mathrm{P}(G|D) &= \frac{\mathrm{P}(D|G)\mathrm{P}(G)}{\mathrm{P}(D)} \\
&= \frac{\mathrm{P}(D|G)\mathrm{P}(G)}{\mathrm{P}(D|G)\mathrm{P}(G) + \mathrm{P}(D|G^c)\mathrm{P}(G^c)} \\
&= \frac{(1)(0.6)}{(1)(0.6) + (0.1)(0.4)} = 0.9375
\end{aligned}$$

$\square$

Example 65. — A medical tests of rare diseases* Medical tests for a certain condition are not perfect. Consider a test that, when performed on an affected person, it comes up positive 95% of the times and yields a “false negative" 5% of the times. When the test is performed on a healthy person the test comes up negative in 99% of the cases and yields a “false positive" in 1% of the cases. If 0.5% of the population actually have the condition, what is the probability that Alice has the condition given that her test came up positive?
We will consider the following events to answer the question.

A: event that Alice has the medical condition.
B: event that Alice tested positive.

Solution: We are interested in $\mathrm{P}(C|P)$. From the definition of conditional probability and the total probability theorem we get

$$\begin{aligned}
\mathrm{P}(A|B) &= \frac{\mathrm{P}(B|A)\mathrm{P}(A)}{\mathrm{P}(B|A)\mathrm{P}(A)+\mathrm{P}(B|A^c)\mathrm{P}(A^c)} \\
&= \frac{0.95\times 0.005}{0.95\times 0.005+0.01\times 0.995} \\
&= 0.323
\end{aligned}$$

This result means that 32.3% of the people who are tested positive actually suffer from the condition!
$\square$

Example 66. — Quality of concrete material* In order to ensure the quality of concrete material used in a reinforced concrete construction, concrete cylinders are collected at random from concrete mixes delivered to the construction site by a mixing plant. Past records of concrete from the same plant show that **80% of concrete mixes are good** or of satisfactory quality. To further ensure that the concrete delivered on site is of good quality, the engineer requires that one cylinder among those collected each day be tested for minimum compressive strength. The test method is not perfect—**its reliability is only 95%**, meaning the probability that a good-quality concrete cylinder will pass the test is 0.95, or that a poor-quality cylinder can pass the test is 0.05. (10 points)
(a) If a concrete cylinder passes the test, find the probability that it is a good-quality concrete delivered on site. (10 points)
(b) Now, suppose the engineer is not satisfied with just testing one cylinder, and requires that a second cylinder be tested. If the second cylinder tested also gave a positive result, find the probability that the concrete is of good quality.
(c) If the third cylinder tested didn't pass the test, find the probability that the concrete is of good quality. (10 points)

Solution: Answer: (a) 0.987 (b) 0.999 (c) 0.981

(a) Define the following events:
G = good quality concrete
T = a concrete cylinder passes the test According to the available information, we have:

$$\begin{aligned} P(G) &= 0.8 \\ P(T|G) &= 0.95 \\ P(T|G^c) &= 0.05 \end{aligned}$$

Then, if a concrete cylinder passes the test, the probability that it is a good-quality concrete delivered on site is updated as follows:

$$\begin{aligned} P(G) &= \frac{P(T|G)P(G)}{P(T|G)P(G)+P(T|G^c)P(G^c)} \\ &= \frac{0.95\times 0.80}{0.95\times 0.80+0.05\times 0.20} \\ &= 0.987 \end{aligned}$$

(b) Here, the second cylinder is tested positive, so now:

$$\begin{aligned}
P(G) &= 0.987 \\
P(G|T) &= \frac{P(T|G)P(G)}{P(T|G)P(G)+P(T|G^c)P(G^c)} \\
&= \frac{0.95\times 0.987}{0.95\times 0.987+0.05\times(1-0.987)} \\
&= 0.999
\end{aligned}$$

(c) The third cylinder is tested negative, so

$$\begin{aligned}
P(G) &= 0.999 \\
P(G|T^c) &= \frac{P(T^c|G)P(G)}{P(T^c|G)P(G)+P(T|G)P(G^c)} \\
&= \frac{0.05\times 0.999}{0.05\times 0.999+0.95\times(1-0.999)} = 0.981
\end{aligned}$$

$\square$

.10 More Problems

Example 67. Suppose the travel time between two major cities A and B by air is 7 or 8 hr if the flight is nonstop; however, if there is one stop, the travel time would be 10, 11, or 12 hr. A nonstop flight between A and B would cost \$1000, whereas with one stop the cost is only \$650. Then, between cities B and C, all flights are nonstop requiring 2 or 3 hours at a cost of \$250. (There is no flight from A to C)
For a passenger wishing to travel from city A to city C,
(a) What is the possibility space or sample space of his travel times from A to B? From A to C?
(b) What is the sample space of his travel cost from A to B?
(c) If T=travel time from city A to city C, and S=cost of travel from A to C, what is the sample space of T and S?

Solution: (a) Sample space of travel time from A to B = $\{7, 8, 10, 11, 12\}$
Sample space of travel time from A to C = $\{9, 10, 11, 12, 13, 14, 15\}$
(b) Sample space of travel cost from A to B = $\{650, 1000\}$
(c) Sample space of T = $\{9, 10, 11, 12, 13, 14, 15\}$
Sample space of S = $\{900, 1250\}$
Sample space of T and S = $\{\{9, 1250\}, \{10, 1250\}, \{11, 1250\}, \{12, 900\}, \{13, 900\}, \{14, 900\}, \{15, 900\}\}$

□

Example 68. Two construction companies $\boldsymbol{a}$ and $\boldsymbol{b}$ are bidding for projects. Define $\boldsymbol{A}$ as the event that Company $\boldsymbol{a}$ wins a bid, and $\boldsymbol{B}$ as the event that Company $\boldsymbol{b}$ wins a bid.
Sketch the Venn diagrams for the sample spaces and their subsets of the following:
(a) Company $\boldsymbol{a}$ is submitting a bid for one project, and Company $\boldsymbol{b}$ is submitting its own bid for another project. (In this case, it is possible for both companies to win their respective bids)
(b) Companies $\boldsymbol{a}$ and $\boldsymbol{b}$ are submitting bids for the same project, and there are also other bidders for the project.
(c) Companies $\boldsymbol{a}$ and $\boldsymbol{b}$ are the only companies submitting competing bids for a single project.

Solution: From Ang and Tang textbook Example 2.9 and Example 2.10 on Page 38-39. In class. $\square$

Example 69. In a process that manufactures aluminum cans, the probability that a can has a flaw on its side is 0.03, the probability that a can has a flaw on its top is 0.05, and the probability that a can has a flaw on both the side and the top is 0.01. What is the probability that it has no flaw?

Solution: $P(\text{on side or on top}) = P(\text{on side}) + P(\text{on top})$ - $P(\text{on side and on top}) = 0.07$
$P(\text{no flaw}) = 1$ - $P(\text{on side or on top}) = 0.93$

$\square$

Example 70. (a) A die is rolled once. What is the probability that the result is even, if it is known that the result is higher than 3?
(b) Two dice are rolled once. what is the sample space of this experiment? what is the probability that the sum is greater than 7?

Solution:
Answer: (a) 2/3 (b) 15/36 (0.417)
A: Result is even
B: Result is higher than 3
C: Sum is greater than 7
(a) $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(\{2,4,6\} \cap \{4,5,6\})}{P(\{4,5,6\})} = \frac{P(\{4,6\})}{P(\{4,5,6\})} = \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}$
(b) Sample space = {{1, 1}, {1, 2}, {1, 3}, {1, 4}, {1, 5}, {1, 6}, {2, 1}, {2, 2}, {2, 3}, {2, 4}, {2, 5}, {2, 6}, {3, 1}, {3, 2}, {3, 3}, {3, 4}, {3, 5}, {3, 6}, {4, 1}, {4, 2}, {4, 3}, {4, 4}, {4, 5}, {4, 6}, {5, 1}, {5, 2}, {5, 3}, {5, 4}, {5, 5}, {5, 6}, {6, 1}, {6, 2}, {6, 3}, {6, 4}, {6, 5}, {6, 6}}
Sample space of the sum = {2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}
$P(C) = \frac{15}{36} = 0.417$ □

Example 71. A team of two engineers, A and B, was assigned to check a set of computations. The two work simultaneously but separately and independently. The probability of engineer A spotting a given error is 0.7, whereas that for B is 0.8.
(a) Suppose there is only one error in the computation. What is the probability that this error will be spotted by this team?
(b) If the error in part (a) was identified, what is the probability that it was discovered by A alone?

Solution:
Answer: (a) 0.94 (b) 0.149
A: A spots the error
B: B spots the error
C: Error spotted
(a) $P(\text{C})$ = 1 - $P(C^c)$ = 1 - $P(A^c)P(B^c)$ = 1 - (1 - 0.7)×(1 - 0.8) = 0.94
(b) $P(AB^c|C) = \frac{P(C|AB^c)}{P(C|AB^c)\cdot P(AB^c)+P(C|A^cB)\cdot P(A^cB)+P(C|AB)\cdot P(AB)+P(C|A^cB^c)\cdot P(A^cB^c)} = 0.149$

□

Example 72. The proportion of people in a given community who have a certain disease is 0.01. A test is available to diagnose the disease. If a person has the disease, the probability that the test will produce a positive signal is 0.98. If a person does not have the disease, the probability that the test will produce a positive signal is 0.02. If a person tests positive, what is the probability that the person actually has the disease?

Solution:
Answer: 0.331
D: The person actually has the disease
+: The tests gives a positive signal
Using Bayes' rule:
$P(D|+) = \frac{P(+|D)P(D)}{P(+|D)P(D)+P(+|D^c)P(D^c)} = \frac{(0.98)(0.01)}{(0.98)(0.01)+(0.02)(1-0.01)} = 0.331$

□

Example 73. — Tornadoes** 100 structures are located in a region where tornado wind force must be considered in its design. Suppose that from the records of tornadoes for the past 200 years, it is estimated that during any 5-year period the probability of having 0, 1 and 2 tornadoes is 0.5, 0.3 and 0.2, respectively. If a tornado occurs, a structure will be damaged with probability $p = 5\%$.

a) if two tornadoes occurred last year, how many structures do you expect to have been damaged?
b) what is the probability the a structure will be damaged in the next five years?
c) how many structures do you expect to be damaged in the next five years?

Solution:

a) If two tornadoes occurred last year, how many structures do you expect to have been damaged?
Let
A_i be the events of having $i = 0, 1, 2$ tornadoes last year, and
D the event that a particular structure was damaged last year.
For any particular structure, it is easier to calculate the probability of no damage given the two tornadoes $(= (1-p)^2)$, so

$$\begin{aligned} \mathrm{P}(D|A_2) &= 1 - \mathrm{P}(D^c|A_2) \\ &= 1 - (1-p)^2 = 0.0975 \end{aligned}$$

and the answer would be $100\mathrm{P}(D|A_2) = 9.75 \rightarrow 10$ structure.

b) what is the probability the a structure will be damaged in the next five years? Let A_i be the events of having $i = 0, 1, 2$ tornadoes in the next five years, and
D the event that a structure will be damaged in the next five years.
Since we don't know the number of tornadoes that will occur, we use the total probability rule:

$$\begin{aligned}
\mathrm{P}(D^c) &= \sum_{i=0}^{2} \mathrm{P}(D^c|A_i)\mathrm{P}(A_i) \\
&= \sum_{i=0}^{2} (1-p)^i \mathrm{P}(A_i) \\
&= (1-p)^0 \mathrm{P}(A_0) + (1-p)^1 \mathrm{P}(A_1) + (1-p)^2 \mathrm{P}(A_2) \\
&= ((1)(0.5)) + ((0.95)(0.3)) + \big((0.95^2)(0.2)\big) = 0.9655
\end{aligned}$$

and the answer is 1-0.9655=0.0345.

c) how many structures do you expect to be damaged in the next five years? 3.45 $\rightarrow$ between 3 and 4 structures.

https://eli.thegreenplace.net/2018/conditional-probability-and-bayes-theorem/

$\square$