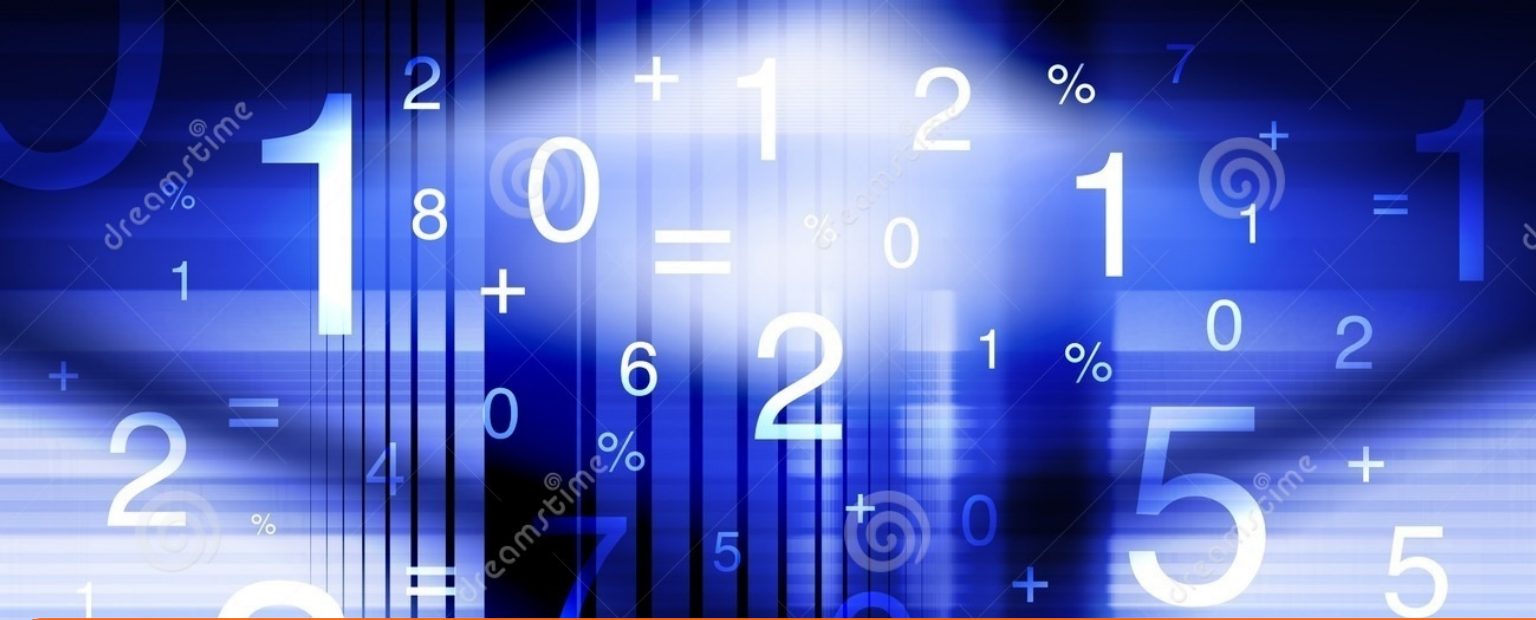
Notes on Probability and Statistics



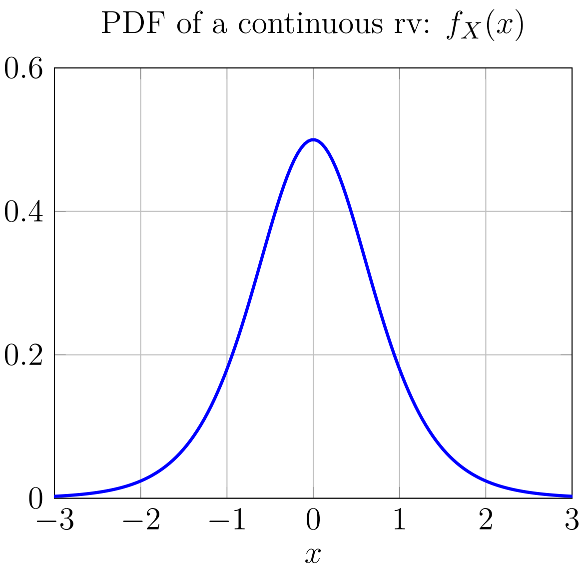
Contentsmkboth CONTENTSCONTENTS

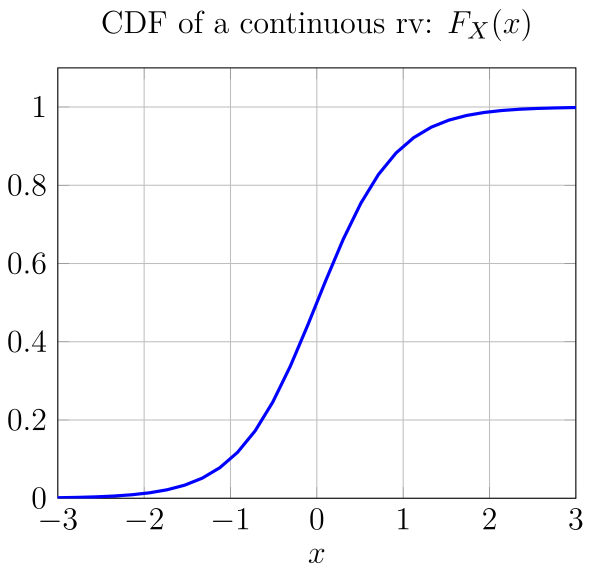
Probability

3. Continuous Random Variables

A **continuous** random variable X takes values in an interval of the real line or all of the real line. Therefore, $F_X(x)$ is continuous (with no jumps), which means that

$\boxed{\mathrm{P}(X = x) = 0 \qquad \text{for all } x}$ and $\boxed{\mathrm{P}(X \leq x) = \mathrm{P}(X < x)}$

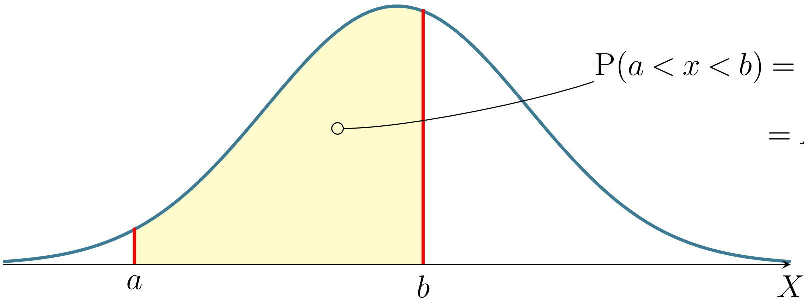




The CDF is still

$$F_X(x) = \mathrm{P}(X \leq x) = \int_{-\infty}^{x} f_X(z)\,dz$$

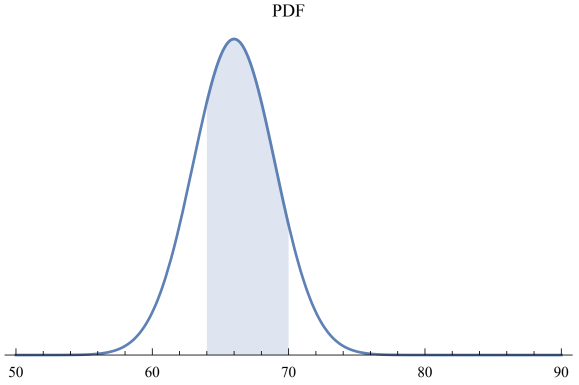
The derivative $\boxed{f_X(x) = F'_X(x)}$ is called **probability density function** (PDF).

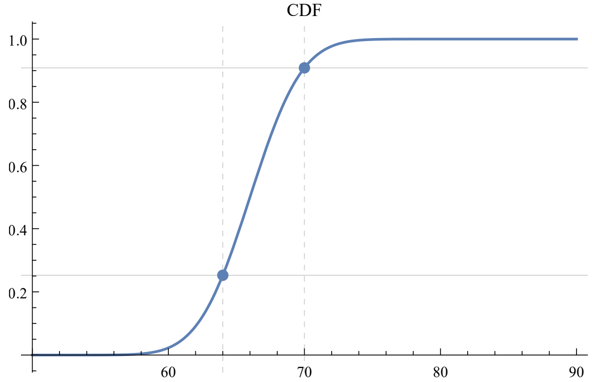


$$\mathrm{P}(a < x < b) = \int_a^b f_X(z)\,dz$$

$$= F_X(b) - F_X(a)$$

→ GeoGebra for interactive probability calculations for the most important distributions.


PDF
50
60
70
80
90


CDF
1.0
0.8
0.6
0.4
0.2
60
70
80
90

Note:

$$F_X(x) = \int_{-\infty}^{x} f_X(z)\, dz.$$

and

$$\int\limits_{-\infty}^{\infty} f_X(x)\,\mathrm{d}x = 1$$

Hence any nonnegative function that integrates to one defines a cdf.

The PDF has units Unlike the PMF of a discrete random variable, the PDF has units:

$$\textbf{units of } f_X(x) = (\textbf{units of } X)^{-1}$$

This means that $f_X(x)$ it's hard to interpret because it depends on the units of measurement. The term $f_X(x)\,\mathrm{d}x$ is meaningful because it represents a probability:

$$\boxed{f_X(x)\,\mathrm{d}x \approx \mathrm{P}(x < X < x + \mathrm{d}x)} \tag{3.1}$$

Expectation For a continuous random variable X the expectation is defined as

$$\mathrm{E}(X) = \int_{-\infty}^{\infty} x f_X(x)\,\mathrm{d}x$$

and for any real function g,

$$\mathrm{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\,\mathrm{d}x$$

Variance The variance formulas are the same as before:

$$\mathrm{V}(X) = \mathrm{E}\left[(X - \mathrm{E}(X))^2\right] = \mathrm{E}\left[X^2\right] - \mathrm{E}(X)^2$$

but using the above definition of expectation.

Example 1. Let X be a continuous variable whose probability density function is

$$f_X(x) = \begin{cases} cx & ; \quad 0 < x < 1 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

a) Find c.
b) Find $\mathrm{E}(X)$.

Solution:

a) Find c.

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} f_X(x)dx \\ &= c\int_0^1 x \ \mathrm{d}x = c/2 \end{aligned}$$

We get $c = 2$

b) Find $\mathrm{E}(X)$.

$$\begin{aligned}\mathrm{E}(X) &= \int_{-\infty}^{\infty} x f_X(x)\,\mathrm{d}x \\ &= \int_0^1 2x^2\,\mathrm{d}x = \frac{2}{3}\end{aligned}$$

$\square$

Example 2. — * The PDF of the random variable X is,

$$f_X(x) = \begin{cases} k(1-x^2+\frac{1}{2}x^3), & 0 < x \leq 2 \\ 0, & otherwise \end{cases}$$

where k is a constant.

(a) Determine the value of k
(b) Determine the **mean value**, **variance** and **coefficient of variation** of X
(c) Determine $P(0.3<X<0.9 \mid X>0.6)$

Solution:
Answer: (a) $k = \frac{3}{4}$ (b) 0.9, 0.39, 0.69 (c) 0.247
(a)

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} f_X(x)dx \\ &= \int_0^2 k(1-x^2+\frac{1}{2}x^3)dx \end{aligned}$$

We get $k = \frac{3}{4}$

(b) Mean

$$
\begin{aligned}
E(X) &= \int_{-\infty}^{\infty} x f_X(x) dx \\
&= \int_0^2 kx(1 - x^2 + \frac{1}{2}x^3) dx \\
&= \int_0^2 \frac{3}{4}x(1 - x^2 + \frac{1}{2}x^3) dx \\
&= 0.9
\end{aligned}
$$

Variance

$$
\begin{aligned}
E(X^2) &= \int_{-\infty}^{\infty} x^2 f_X(x)dx \\
&= \int_0^2 kx^2(1-x^2+\frac{1}{2}x^3)dx \\
&= \int_0^2 \frac{3}{4}x^2(1-x^2+\frac{1}{2}x^3)dx \\
&= 1.2
\end{aligned}
$$

$$
\begin{aligned}
V(X) &= E(X^2)-[E(X)]^2 \\
&= 1.2-0.9^2 \\
&= 0.39
\end{aligned}
$$

Coefficient of Variation

$$
\begin{aligned}
\delta_X &= \frac{\sqrt{V(X)}}{E(X)} \\
&= 0.69
\end{aligned}
$$

(c)

$$
\begin{aligned}
P(0.3 < X < 0.9 | X > 0.6) &= \frac{P(0.6 < X < 0.9)}{P(X > 0.6)} \\
&= \frac{\int_{0.6}^{0.9} f_X(x)dx}{\int_{0.6}^{\infty} f_X(x)dx} \\
&= \frac{0.554 - 0.408}{1 - 0.408} \\
&= 0.247
\end{aligned}
$$

$\square$

Example 3. Let X be a continuous variable whose probability density function is

$$f_X(x) = \begin{cases} c\left(4x - 2x^2\right) & ; \quad 0 < x < 2 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

a) Find c.
b) Find $\mathrm{P}(X > 1)$.

Solution:
1. As $f_X(x)$ is a probability density function,

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} f_X(x)\,\mathrm{d}x \\ &= \int_{0}^{2} c\left(4x^2 - 2x^2\right)\mathrm{d}x \\ &= \frac{8c}{3} \end{aligned}$$

Therefore,

$$c = \frac{3}{8}$$

2.

$$\begin{aligned} \mathrm{P}(X > 1) &= \int\limits_{1}^{\infty} f_X(x)\,\mathrm{d}x \\ &= \int\limits_{1}^{2} \frac{3}{8}\left(4x - 2x^2\right)\mathrm{d}x \\ &= \frac{1}{2} \end{aligned}$$

$\square$

Example 4. — Cauchy distribution Let X be a continuous variable whose probability density function is

$$f_X(x) = \frac{c}{1+x^2} \qquad -\infty < x < \infty$$

a) Find c.
b) Find $F_X(x)$.
c) Find $\mathrm{V}(X)$.

Solution:
a) As $f_X(x)$ is a probability density function,

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} f_X(x)\,\mathrm{d}x \\ &= c\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,\mathrm{d}x \\ &= c\pi \end{aligned}$$

Therefore,

$$c = \frac{1}{\pi}$$

b) $F_X(x) = \int_{-\infty}^{x} \frac{1}{\pi\left(y^2+1\right)}\, dy = \pi\left(\tan^{-1}(x) + \frac{\pi}{2}\right)$

c) $\mathrm{E}(X) = \int_{-\infty}^{\infty} x \cdot \frac{1}{\pi\left(x^2+1\right)}\, dx$ does not converge, and nor does $\mathrm{E}\left(X^2\right)$, or the variance $\mathrm{V}(X)$: the Cauchy distribution is said to be “pathological”.

$\square$

Example 5. — Exponential distribution. The amount of time in hours that a computer functions before breaking down has the distribution

$$f_X(x) = \begin{cases} \lambda e^{-\frac{x}{100}} & ; \quad x \geq 0 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

What is the probability that the computer functions for more than 50 but less than 150 hours?

Solution:

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} f_X(x)\,\mathrm{d}x \\ &= \lambda \int_{0}^{\infty} e^{-\frac{x}{100}}\,\mathrm{d}x \\ &= 100\lambda \end{aligned}$$

Therefore,

$$\lambda = \frac{1}{100}$$

Therefore,

$$\begin{aligned}
\mathrm{P}(50 < x < 150) &= \int\limits_{50}^{150} \lambda e^{-\frac{x}{100}} \,\mathrm{d}x \\
&= \int\limits_{50}^{150} \frac{1}{100} e^{-\frac{x}{100}} \,\mathrm{d}x \\
&\approx 0.384
\end{aligned}$$

$\square$

Example 6. — Uniform distribution. The probability density function of X is given by the Uniform distribution in (0, 1):

$$f_X(x) = \begin{cases} 1 & ; \quad 0 \leq x \leq 1 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

Find $\mathrm{E}\left[e^X\right]$.

Solution:

$$\begin{aligned} \mathrm{E}\left[e^X\right] &= \int_{-\infty}^{\infty} e^x f_X(x)\,\mathrm{d}x \\ &= \int_0^1 e^x\,\mathrm{d}x \\ &= e - 1 \end{aligned}$$

□

3.1 Joint Continuous Variables

Joint CDF and PDF X and Y are said to be jointly continuous if there exists a function $f_{X,Y}(x,y)$ defined for all real x and y, such that

$$\begin{aligned}\mathrm{F}_{X,Y}(x,y) &= \mathrm{P}(-\infty \leq X \leq x, -\infty \leq Y \leq y) \\ &= \int_{-\infty}^{y}\int_{-\infty}^{x} f_{X,Y}(a,b)\,\mathrm{d}a\,\mathrm{d}b\end{aligned}$$

Therefore: $f_{X,Y}(x,y) = \frac{\partial^2}{\partial x \partial y} F(x,y)$.
Note:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y = 1$$

Probability of events. An event C is any subset (area or region) of the $X-Y$ plane, and

$$\mathrm{P}\left((X,Y) \in C\right) = \iint\limits_{(x,y)\in C} f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y \tag{3.2}$$

Marginal PDF The PDF of a single random variable is called marginal PDF:

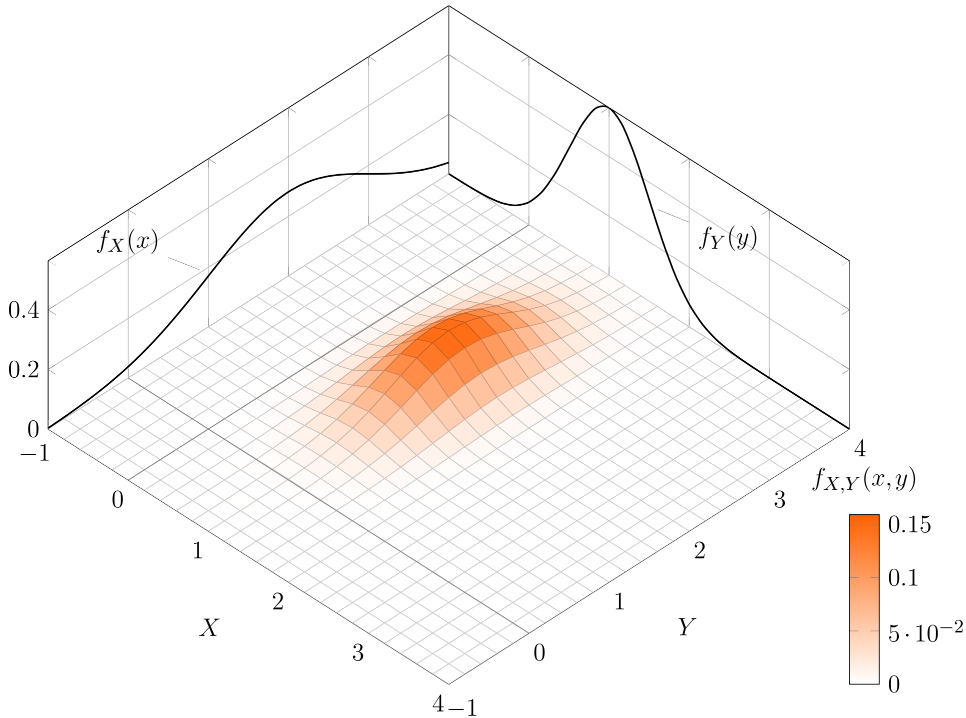
$$f_X(x) = \int_{y=-\infty}^{\infty} f_{X,Y}(x,y)\,\mathrm{d}y \qquad \text{and} \qquad f_Y(y) = \int_{x=-\infty}^{\infty} f_{X,Y}(x,y)\,\mathrm{d}x$$

Conditional probability density functions For two continuous random variables X and Y:

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

and the cumulative distribution function is

$$\mathrm{F}_{X|Y}(x) = \int_{-\infty}^{x} f_{X|Y}(x)\,\mathrm{d}x$$



$f_X(x)$
$f_Y(y)$
0.4
0.2
0
−1
0
1
2
3
4
X
−1
0
1
2
3
4
Y
$f_{X,Y}(x,y)$
0.15
0.1
$5 \cdot 10^{-2}$
0

Independence If X and Y are continuous, then

$$f_{X,Y}(x,y) = f_X(x)f_Y(y)$$

for all (x,y), if and only if X and Y are independent.

Expectation

$$\mathrm{E}\left(g(X,Y)\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

$$\mathrm{E}\left(g(X,Y) \mid Y = y\right) = \int_{-\infty}^{\infty} g(x,y)\, f_{X|Y}(x)\,\mathrm{d}x$$

Notice that:

$$\begin{aligned}
\mathrm{E}[Y] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y\, f_{X,Y}(x, y)\, \mathrm{d}x\, \mathrm{d}y \\
&= \int_{-\infty}^{\infty} y \left(\int_{-\infty}^{\infty} f_{X,Y}(x, y)\, \mathrm{d}x \right) \mathrm{d}y \\
&= \int_{-\infty}^{\infty} y f_Y(y)\, \mathrm{d}y
\end{aligned}$$

Covariance The formula for covariance is identical to the discrete case,

$$\begin{aligned}\mathrm{Cov}(X,Y) &= \mathrm{E}\left[(X-\mathrm{E}(X))(Y-\mathrm{E}(Y))\right]\\ &= \mathrm{E}(XY)-\mathrm{E}(X)\mathrm{E}(Y)\end{aligned}$$

but using the definition of expectation above.

Example 7. — * Two friends meet? Two friends try to meet at a certain place between 5 pm and 6 pm. Each person arrives at a time uniformly distributed in the time - interval independently of each other and stays for **20 minutes**.

Find the probability that they meet.

Solution: We have:

$$\begin{aligned} X &\sim \mathrm{U}(0,60) \\ Y &\sim \mathrm{U}(0,60) \end{aligned}$$

with the Uniform distribution in (0, 60):

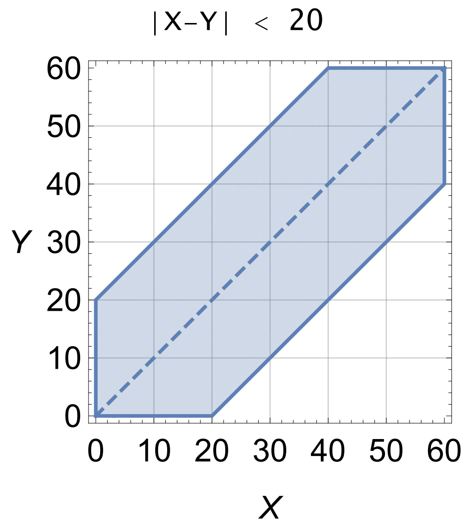
$$f_X(x) = \begin{cases} 1/60 & ; \quad 0 \le x \le 60 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

and therefore by independence

$$\begin{aligned} f_{X,Y}(x,y) &= f_X(x) f_Y(y) \\ &= \begin{cases} 1/60^2 & ; \quad 0 \le x, y \le 60 \\ 0 & ; \quad \text{otherwise} \end{cases} \end{aligned}$$

The event of interest C is $|X - Y| \leq 20$. From the figure we can see that

$$
\begin{aligned}
\mathrm{P}\left((X,Y) \in C\right) &= \iint\limits_{(x,y)\in C} f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \iint\limits_{(x,y)\in C} \frac{1}{60^2}\,\mathrm{d}x\,\mathrm{d}y \\
&= \text{area of } C/60^2 \\
&= 0.55
\end{aligned}
$$



□

Example 8. Let

$$X \sim \mathrm{U}(0,1)$$
$$Y \sim \mathrm{U}(0,1)$$

Calculate the probability density function of $X+Y$

Solution: Let's compute the CDF of $X+Y$ and then take derivatives to obtain the PDF. For the CDF, the event of interest C is $X+Y \leq t$. From the graphical representation of this event in the X-Y plane we can see that

$$\begin{aligned}
\mathrm{F}_{X+Y}(t) &= \mathrm{P}(X+Y \leq t) \\
&= \iint_{(x,y)\in C} f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y \qquad \text{but } f_{X,Y}(x,y) = 1, \text{ and from the figure seen in class:} \\
&= \begin{cases} \frac{t^2}{2} & ; \quad 0 \leq t \leq 1 \\ 1 - \frac{(2-t)^2}{2} & ; \quad 1 \leq t \leq 2 \\ 0 & ; \quad \text{otherwise} \end{cases}
\end{aligned}$$

Therefore, taking derivatives

$$f_{X+Y}(t) = \begin{cases} t & ; \quad 0 \leq t \leq 1 \\ 2-t & ; \quad 1 < t < 2 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

$\square$

Example 9. — * An accident occurs at a point X that is uniformly distributed on a road of length L. At the time of the accident, and ambulance is at a location Y that is also uniformly distributed on the same road.
Assuming that X and Y are independent, find the expected distance $|X-Y|$ between the point of occurrence of the accident, and the position of the ambulance.

Solution:

$$f_X(x) = \begin{cases} \frac{1}{L} & ; \quad 0 < x < L \\ 0 & ; \quad \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{1}{L} & ; \quad 0 < y < L \\ 0 & ; \quad \text{otherwise} \end{cases}$$

As the variables are independent,

$$\begin{aligned} f_{X,Y}(x,y) &= f_X(x) f_Y(y) \\ &= \begin{cases} \frac{1}{L^2} & ; \quad 0 < x < L, 0 < y < L \\ 0 & ; \quad \text{otherwise} \end{cases} \end{aligned}$$

Therefore,

$$
\begin{aligned}
\mathrm{E}\left[|X-Y|\right] &= \frac{1}{L^2}\int_0^L\int_0^L |x-y|\,\mathrm{d}y\,\mathrm{d}x \\
&= \frac{1}{L^2}\int_0^L \left(\int_0^x (x-y)\,\mathrm{d}y + \int_x^L (y-x)\,\mathrm{d}y\right)\mathrm{d}x \\
&= \frac{1}{L^2}\int_0^L \left(\frac{L^2}{2} + x^2 - xL\right)\mathrm{d}x \\
&= \frac{L}{3}
\end{aligned}
$$

□

This example shows how random variables can have a $\mathrm{Cov}(X,Y)=0$ but still be dependent:

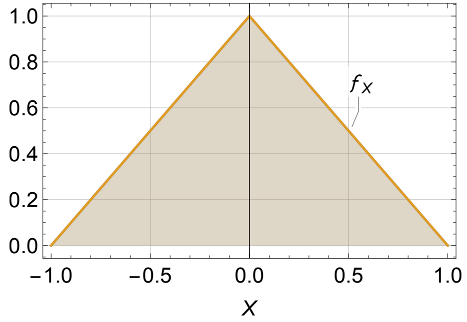
Example 10. Find Cov(X,Y) for $Y = X^2$ where X has a Triangular Distribution in (-1,1).

Solution: Let

$$\begin{cases} X \sim Triang(-1,1) \\ Y = X^2 \end{cases}$$

with

$$f_X(x) = \begin{cases} 1+x & -1 \leq x \leq 0 \\ 1-x & 0 < x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$



Therefore,

$$\begin{aligned}
\mathrm{E}(X) &= 0 \\
\mathrm{E}(Y) &= \mathrm{E}\left(X^2\right) = \int_{-1}^{1} x^2 f_X(x)\,\mathrm{d}x = \int_{-1}^{0} x^2(1+x)\,\mathrm{d}x + \int_{0}^{1} x^2(1-x)\,\mathrm{d}x \\
&= \int_{-1}^{0} x^2\,\mathrm{d}x + \int_{0}^{1} x^2\,\mathrm{d}x = 2/3 \\
\mathrm{E}(XY) &= \mathrm{E}\left(X^3\right) = \int_{-1}^{1} x^3 f_X(x)\,\mathrm{d}x = \int_{-1}^{0} x^3(1+x)\,\mathrm{d}x + \int_{0}^{1} x^3(1-x)\,\mathrm{d}x \\
&= \int_{-1}^{0} x^3\,\mathrm{d}x + \int_{0}^{1} x^3\,\mathrm{d}x = 0
\end{aligned}$$

Therefore, $\mathrm{Cov}(X,Y) = 0$ - $0 = 0$ $\square$

Example 11. The joint density function of X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} 2e^{-x}e^{-2y} & ; \quad 0 < x < \infty, 0 < y < \infty \\ 0 & ; \quad \text{otherwise} \end{cases}$$

Compute

1. $\mathrm{P}(X > 1, Y < 1)$
2. $\mathrm{P}(X < Y)$
3. $\mathrm{P}(X < a)$

Solution:

1.

$$
\begin{aligned}
\mathrm{P}(X > 1, Y < 1) &= \int_{-\infty}^{1} \int_{1}^{\infty} f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{0}^{1} \int_{1}^{\infty} 2e^{-x} e^{-2y}\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{0}^{1} 2e^{-2y} \left. -e^{-x} \right|_{x=1}^{x=\infty} \mathrm{d}y \\
&= e^{-1} \int_{0}^{1} 2e^{-2y}\,\mathrm{d}y \\
&= e^{-1} \left(1 - e^{-2}\right)
\end{aligned}
$$

2.

$$
\begin{aligned}
\mathrm{P}(X<Y) &= \iint\limits_{(x,y):x<y} f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \iint\limits_{(x,y):x<y} 2e^{-x}e^{-2y}\,\mathrm{d}x\,\mathrm{d}y \\
&= \int\limits_0^\infty \int\limits_0^y 2e^{-x}e^{-2y}\,\mathrm{d}x\,\mathrm{d}y \\
&= \int\limits_0^\infty 2e^{-2y}\left(1-e^{-y}\right)\mathrm{d}y \\
&= \int\limits_0^\infty 2e^{-2y}\,\mathrm{d}y - \int\limits_0^\infty 2e^{-3y}\,\mathrm{d}y \\
&= 1-\frac{2}{3} \\
&= \frac{1}{3}
\end{aligned}
$$

3.

$$
\begin{aligned}
\mathrm{P}(X < a) &= \int_{-\infty}^{a} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_{0}^{a} \int_{0}^{\infty} 2e^{-2y} e^{-x}\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_{0}^{a} e^{-x}\,\mathrm{d}x \\
&= 1 - e^{-a}
\end{aligned}
$$

$\square$

Example 12. Consider the bivariate density function

$$f(x,y) = \frac{12}{7}(x^2 + xy), \quad 0 \leq x,y \leq 1.$$

Find the probability that $X > Y$.

Solution: The desired probability can be found by integrating f over the region $A = \{(x, y) | 0 \leq y \leq x \leq 1\}$. Note that A is not a rectangle, so we use (**??**):

$$P(X > Y) = \frac{12}{7} \int_0^1 \int_0^x (x^2 + xy)\ dy\ dx = \frac{9}{14}.$$

$\square$

Example 13. — * The joint density function of X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} e^{-(x+y)} & ; \quad 0 < x < \infty, 0 < y < \infty \\ 0 & ; \quad \text{otherwise} \end{cases}$$

a) Are X and Y independent? Explain why.
b) Find the density function of the random variable $Z = X/Y$.
c) Find the expected value of the function $g(X,Y) = XY$. *Hint*: $\int xe^{-x} = -(x+1)e^{-x}$.

Solution:

a) Are X and Y independent? Explain why. Yes because the joint distribution is the product of the marginals:

$$f_{X,Y}(x,y) = f_X(x) f_Y(y)$$

$$f_X(x) = \int_0^\infty e^{-(x+y)} dy = e^{-x} \int_0^\infty e^{-y} dy = e^{-x}$$

So indeed,

$$e^{-(x+y)} = e^{-x}e^{-y}$$

b) Find the density function of the random variable $Z = X/Y$.

$$
\begin{aligned}
\mathrm{F}_{\frac{X}{Y}}(a) &= \mathrm{P}\left(\frac{X}{Y} \leq a\right) \\
&= \iint\limits_{(x,y):\frac{x}{y}\leq a} f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \iint\limits_{(x,y):\frac{x}{y}\leq a} e^{-(x+y)}\,\mathrm{d}x\,\mathrm{d}y \\
&= \int\limits_0^\infty \int\limits_0^{ay} e^{-(x+y)}\,\mathrm{d}x\,\mathrm{d}y \\
&= \int\limits_0^\infty \left(1 - e^{-ay}\right) e^{-y}\,\mathrm{d}y \\
&= -e^{-y} + \left.\frac{e^{-(a+1)y}}{a+1}\right|_0^\infty = 1 - \frac{1}{a+1}
\end{aligned}
$$

Therefore,

$$\begin{aligned} f_{\frac{X}{Y}}(a) &= \frac{\mathrm{dF}_{\frac{X}{Y}}(a)}{\mathrm{d}a} \\ &= \frac{1}{(a+1)^2} \end{aligned}$$

c) Find the expected value of the function $g(X,Y) = XY$. *Hint*: $\int xe^{-x} = -(x+1)e^{-x}$.
Since X, Y are *independent*:

$$\mathrm{E}(XY) = \mathrm{E}(X)\,\mathrm{E}(Y) = 1 \times 1$$

$\square$

Example 14. — * Two random variables X and Y have the following joint PDF:

$$f_{X,Y}(x,y) = \begin{cases} kx & ; \quad 0 < x < 1, 0 < y < x \\ 0 & ; \quad \text{otherwise} \end{cases}$$

Determine:
(a) the value of k
(b) the conditional and marginal PDF of X
(c) the conditional and marginal PDF of Y
(d) the standard deviation of X
(e) the standard deviation of Y
(f) the correlation coefficient between X and Y
(g) the mean and variance of the function $g(X,Y) = | X - Y |$

Solution: Answer: (a) 3 (d) 0.193 (e) 0.244 (f) 0.397 (g) 0.375, 0.0594

(a)

$$\iint kx \mathrm{d}x \mathrm{d}y = 1$$

We get $k = 3$

(b) Marginal PDF of X:

$$\int_0^x 3x dy = 3x^2, 0 < x < 1$$

Marginal PDF of Y:

$$\int_y^1 3x dx = \frac{3}{2}(1 - y^2), 0 < y < 1$$

(c) Conditional PDF of X:

$$\begin{aligned} f_{X|Y} &= \frac{3x}{\frac{3}{2}(1-y^2)} \\ &= \frac{2x}{1-y^2}, 0 < x < 1 \end{aligned}$$

Conditional PDF of Y:

$$\begin{aligned} f_{Y|X} &= \frac{3x}{3x^2} \\ &= \frac{1}{x}, 0 < y < 1 \end{aligned}$$

(d)

$$\begin{aligned} E(X) &= \int_0^1 x \cdot 3x^2 dx \\ &= \frac{3}{4} \end{aligned}$$

$$\begin{aligned} E(X^2) &= \int_0^1 x^2 \cdot 3x^2 dx \\ &= \frac{3}{5} \end{aligned}$$

$$\begin{aligned} \sigma_X &= \sqrt{E(X^2) - E^2(X)} \\ &= 0.193 \end{aligned}$$

(e)

$$
\begin{aligned}
E(Y) &= \int_0^1 y(1-y^2)dy \\
&= \frac{3}{8}
\end{aligned}
$$

$$
\begin{aligned}
E(Y^2) &= \int_0^1 y^2(1-y^2)dy \\
&= \frac{1}{5}
\end{aligned}
$$

$$
\begin{aligned}
\sigma_Y &= \sqrt{E(Y^2) - E^2(Y)} \\
&= 0.244
\end{aligned}
$$

(f)

$$
\begin{aligned}
E(XY) &= \iint xy \cdot 3x \mathrm{d}x \mathrm{d}y \\
&= \frac{3}{10}
\end{aligned}
$$

$$
\begin{aligned}
\rho &= \frac{Cov(X,Y)}{\sigma_X \sigma_Y} \\
&= \frac{E(XY) - E(X)E(Y)}{\sigma_X \sigma_Y} \\
&= 0.397
\end{aligned}
$$

(g) Because $0 < y < x$, $|X - Y| = x - y$

$$
\begin{aligned}
E[|X-Y|] &= E(X-Y) \\
&= \frac{3}{8} \\
E[|X-Y|^2] &= E[(X-Y)^2] \\
&= \frac{1}{5}
\end{aligned}
$$

$$
\begin{aligned}
V[|X-Y|] &= E[|X-Y|^2] - E^2[|X-Y|] \\
&= \frac{19}{320} \\
&= 0.0594
\end{aligned}
$$

$\square$

Example 15.

$$f_{X,Y}(x,y) = \begin{cases} \dfrac{e^{-\frac{x}{y}}e^{-y}}{y} & ; \quad 0 < x < \infty, 0 < y < \infty \\ 0 & ; \quad \text{otherwise} \end{cases}$$

Find $\mathrm{P}(X > 1 | Y = y)$.

Solution:

$$\begin{aligned} f_{X|Y}(x) &= \frac{f_{X,Y}(x,y)}{f_Y(y)} \\ &= \frac{\frac{e^{-\frac{x}{y}}e^{-y}}{y}}{e^{-y}\int\limits_0^\infty \left(\frac{1}{y}\right) e^{-\frac{x}{y}}\,\mathrm{d}x} \\ &= \frac{e^{-\frac{x}{y}}}{y} \end{aligned}$$

Therefore,

$$
\begin{aligned}
\mathrm{P}(X > 1|Y = y) &= \int_{1}^{\infty} \frac{1}{y} e^{-\frac{x}{y}} \, \mathrm{d}x \\
&= e^{-\frac{1}{y}}
\end{aligned}
$$

$\square$

3.1.1 Exercises

1. Let the rv's X and Y have the joint pdf given below: f(x,y)=kxy^2, for $0 \leq x \leq 2,\ x \leq y \leq 3$.
   2. Find the constant k.
   3. Find the marginal pdf's of X and Y.
   4. Are X and Y independent?

Let the rv's X and Y have the joint pdf given below:

$$f(x,y) = \begin{cases} 2e^{-x-y} & 0 \leq x \leq y < \infty \\ 0 & otherwise \end{cases}$$

1. Find $P(X+Y \leq 3)$.
2. Find the marginal pdf's of Y and X.
3. Are X and Y independent? Justify your answer.

Let X be the force applied to a randomly selected beam, and Y the time to failure of the beam. Suppose that X is uniformly distributed between 500 and 600 pounds. Suppose also that the conditional pdf of Y given that a force $X = x$ is applied is zero for negative y and f$_{Y|X=x}(y) = \lambda(x)e^{-\lambda(x)y}$, for $y > 0, where \lambda(x) = 0.02x - 9.999$.

Find the joint distribution of (X,Y).

Find the expected time to failure of a randomly selected beam when the force applied is $X = 580$. (*Hint:* Use the formula for the mean value of an exponential random variable.)

A type of steel has microscopic defects which are classified on continuous scale from 0 to 1, with 0 the least sever and 1 the most sever. This is called the defect index. Let X and Y be the static force at failure and the defect index for a particular type of structural member made of this steel. For a member selected at random, these are jointly distributed random variables with joint pdf

$$f(x,y) = \begin{cases} 24x & \text{if } 0 \le y \le 1-2x \text{ and } 0 \le x \le .5 \\ 0 & \text{otherwise} \end{cases}$$

1. Draw the support of this pdf, i.e. the region of (x,y) values where $f(x,y) > 0$.
2. Are X and Y independent? Answer this question without first computing the marginal pdfs of X and Y. Justify your answer.
3. Find each of the following: f_X, f_Y, $E(X)$, and $E(Y)$.
4. Find the conditional pdf of Y given $X = x$.
5. Use the conditional pdf found above to calculate $E(Y|X = 0.3)$.

John and his trainer Yvonne have agreed to meet between 6 A.M. and 8 A.M. for a workout but will aim for 6 A.M. Let $X = \#$ of hours that John is late, and $Y = \#$ of hours that Yvonne is late. Suppose that the joint distribution of X and Y is

$$f(x,y) = \begin{cases} \frac{1}{4} & 0 \le x \le 2, 0 \le y \le 2 \\ 0 & \text{elsewhere} \end{cases}$$

1. Determine the marginal probability density function of X. Do the same for Y. [If you can, guess the marginal pdf of Y without any additional calculations.]
2. Compute $E(X)$ and $\text{Var}(X)$. Do the same for Y. [If you can, guess the $E(Y)$ and $\text{Var}(Y)$ without any additional calculations.]
3. Are X and Y independent? Justify your answer.